Applied Mori theory of the moduli space of stable pointed rational curves

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To my grandparents, Wilbur and Cleta.

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Abstract

We investigate questions motivated by Mori's program for the moduli space of stable pointed rational curves, $\overline{M}_{0,n}$. In particular, we study the nef cone of $\overline{M}_{0,n}$ (Chapter 2), the Cox ring of $\overline{M}_{0,n}$ (Chapter 3), and the cone of movable curves of $\overline{M}_{0,6}$ (Chapter 4).

In Chapter 2, we prove Fulton's conjecture for $\overline{M}_{0,n}$, $n \leq 7$, which states that any divisor on $\overline{M}_{0,n}$ non-negatively intersecting all members of a distinguished, finite collection of curves, called F-curves, is linearly equivalent to an effective integral sum of boundary divisors. As a corollary, it follows that a divisor on $\overline{M}_{0,n}$ is nef if and only if the divisor intersects all F-curves non-negatively. By duality, we thus recover Keel and McKernan's result that the F-curves generate the closed cone of curves of $\overline{M}_{0,n}$ for $n \leq 7$, but with methods that do not rely on negativity properties of the canonical bundle that fail for higher n.

Chapter 3 initiates a study of relations among generators of the Cox ring of $\overline{M}_{0,n}$. We first prove a 'relation-free' result that exhibits polynomial subrings of $\mathrm{Cox}(\overline{M}_{0,n})$ in boundary section variables. If $\mathcal{G}_{bnd}$ is a collection of boundary divisor sections, one for each boundary divisor, and if $\phi : \mathbb{C}[\mathcal{G}_{bnd}] \to \mathrm{Cox}(\overline{M}_{0,n})$ is the homomorphism sending a boundary section to its image in the Cox ring, then these polynomial subrings in $\mathrm{Cox}(\overline{M}_{0,n})$ are the image under ϕ of subrings having trivial intersection with the kernel of ϕ, i.e. they meet the ideal of relations trivially. In the opposite direction, we exhibit multidegrees in $\mathrm{Cox}(\overline{M}_{0,n})$ such that the corresponding graded parts of $\mathbb{C}[\mathcal{G}_{bnd}]$ meet the ideal of relations non-trivially, hence giving 'relation-full' collections in the Cox ring.

In Chapter 4, we study the so-called *complete intersection cone* of the threefold $\overline{M}_{0,6}$. For a smooth projective variety X, this cone is defined as the closure of curve classes obtained as intersections of $\dim(X) - 1$ very ample divisors. The complete intersection cone is contained in the cone of movable curves, which, by results of Boucksom, Demailly, Păun, and Peternell, is dual to the cone of pseudoeffective divisors. We show that, for a series of toric birational models for $\overline{M}_{0,6}$ related to the Kapranov blow-up construction, the complete intersection and movable cones coincide, while for $\overline{M}_{0,6}$, there is strict containment of these cones.

Key words: Algebraic geometry, moduli spaces of curves, birational geometry, Mori theory, toric varieties, combinatorics

Zusammenfassung

Diese Dissertation befasst sich mit Fragen über den Modulraum $\overline{M}_{0,n}$ der stabilen punktierten rationalen Kurven, die durch das Mori-Programm motiviert sind. Insbesondere studieren wir den nef-Kegel von $\overline{M}_{0,n}$ (Chapter 2), den Cox-Ring von $\overline{M}_{0,n}$ (Chapter 3), und den Kegel der beweglichen Kurven von $\overline{M}_{0,6}$ (Chapter 4).

In Kapitel 2 beweisen wir Fultons Vermutung für $\overline{M}_{0,n}$, $n \leq 7$. Diese Vermutung besagt, dass ein Divisor, der alle Elemente einer gewissen endlichen Kollektion von Kurven, sogenannte F-Kurven, mit nichtnegativer Multiplizität schneidet, als effektive ganzzahlige Linearkombination von Randdivisoren dargestellt werden kann. Als Korollar folgt, dass ein Divisor genau dann nef ist, wenn die entsprechende Schnittmultiplizität mit jeder F-Kurve nichtnegativ ist. Mittels Dualität bekommen wir dadurch einen neuen Beweis des Resultats von Keel und McKernan, dass die F-Kurven für $n \leq 7$ den Kegel der Kurven von $\overline{M}_{0,n}$ erzeugen, jedoch mit Methoden, die unabhängig von Negavititätseigenschaften des kanonischen-Bündels sind, welche für größeres n nicht mehr stimmen.

Kapitel 3 beginnt ein Studium der Relationen zwischen erzeugenden Elementen des Cox-Rings von $\overline{M}_{0,n}$. Wir beweisen zuerst einen "Relationenfreiheitssatz", der in $\mathrm{Cox}(\overline{M}_{0,n})$ polynomiale Unterringe indentifiziert. Ist $\mathcal{G}_{bnd}$ eine Kollektion von Schnitten von Randdivisoren, einer für jeden Randdivisor, und bezeichnet $\phi : \mathbb{C}[\mathcal{G}_{bnd}] \to \mathrm{Cox}(\overline{M}_{0,n})$ den Homomorphismus, der ein zu einem Randdivisor gehöriges Element aus $\mathbb{C}[\mathcal{G}_{bnd}]$ auf sein Bild in $\mathrm{Cox}(\overline{M}_{0,n})$ schickt, dann ist jeder dieser polynomialen Unterringe in $\mathrm{Cox}(\overline{M}_{0,n})$ das Bild unter ϕ von einem Unterring, dessen Schnit mit dem Kern von ϕ trivial ist, das heißt, der Unterring schneidet das Ideal der Relationen trivial. In der anderen Richtung geben wir Multigrade in $\mathrm{Cox}(\overline{M}_{0,n})$ an, sodass der zuhegörige graduierte Tel von $\mathbb{C}[\mathcal{G}_{bnd}]$ das Ideal der Relationen nichttrivial schneidet und erhalten so in Resultat über "volle Relationen".

In Kapitel 4 studieren wir den sogennanten *Kegel der vollständigen Durchschnitte* der 3-Fläche $\overline{M}_{0,6}$. Für eine glatte projektive Varietät X ist dieser Kegel als Abschluss der Kurven definiert, die sich als Durchshnitt von $\dim(X) - 1$ sehr amplen Divisoren darstellen lassen. Der Kegel der vollständigen Durchschnitte ist ein Unterkegel des Kegels der beweglichen Kurven. Nach Ergebnissen von Boucksom, Demailly, Păun, und Peternell ist der bewegliche Kegel dual zum Kegel der pseudoeffektiven Divisoren. Wir beweisen für eine Reihe von torischen birationalen Modellen von $\overline{M}_{0,6}$, dass der Kegel der vollständigen Durchschnitte und der bewegliche Kegel übereinstimmen, während für $\overline{M}_{0,6}$ die Inklusion echt ist.

Schlagworte: Algebraische Geometrie, Modelräume von Kurven, Birationelle Geometrie, Mori-Theorie, Torische Varietäten, Kombinatorik

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1 Introduction

The moduli space of stable pointed rational curves, $\overline{M}_{0,n}$, can be approached by many paths. Introduced by Grothendieck in [28], $\overline{M}_{0,n}$ has been studied via the minimal model program, geometric invariant theory, operads, and classical geometry of Veronese curves. The starting point of most any investigation of this moduli space, however, is Knudsen's result in [41] that $\overline{M}_{0,n}$ is a smooth, projective variety.

Smoothness and compactness (which follow from $\overline{M}_{0,n}$ being projective) imply that Poincaré duality holds, so the intersection product gives a perfect pairing between homology groups of cycles of complementary dimension. In particular, two algebraic cycles are numerically equivalent if and only if they are homologically equivalent. By results of Keel in [38], the Chow and homology rings of $\overline{M}_{0,n}$ are isomorphic, so all homology classes can be represented by algebraic subvarieties, and thus numerical, homological, and rational equivalence all coincide for $\overline{M}_{0,n}$.

A second consequence of Knudsen's result is that all numerical properties of algebraic codimension k-cycles are encoded in finite-dimensional vector spaces $N^k(\overline{M}_{0,n})_{\mathbb{R}}$ (see Definition 1.1.1). The cones of curves and divisors of primary interest in birational geometry, such as the pseudoeffective, moving, and nef cones of divisors, and the movable and closed cones of curves, are convex cones in the vector spaces $N^1(\overline{M}_{0,n})_{\mathbb{R}}$ and $N_1(\overline{M}_{0,n})_{\mathbb{R}}$, respectively.

A third essential characteristic of $\overline{M}_{0,n}$ is its stratification by topological type (see Section 1.2.3). This stratification prompted Fulton to ask if the k-strata of $\overline{M}_{0,n}$ behaved like those of a toric variety, even though, except for $n \leq 4$, $\overline{M}_{0,n}$ is not toric.

These three properties suggest an approach to $\overline{M}_{0,n}$ that complements the more standard tools from birational geometry, intersection theory, and classical projective geometry with methods from toric and convex geometry. In the present work, we employ this more combinatorial tack to answer questions that arise in Mori's program of classifying algebraic varieties up to birational equivalence. For $\overline{M}_{0,n}$, the birational type is easy to determine: since it is a compactification of the moduli space of n distinct points on the projective line modulo projective equivalence, which is isomorphic to the Cartesian product of $n-3$ copies of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, $\overline{M}_{0,n}$ is birational to $\mathbb{P}^{n-3}$ (see, for example, [42], 1.1). Instead, we apply the combinatorial approach mentioned above to study objects that are central to Mori's program. In particular, we answer questions about the nef cone of $\overline{M}_{0,n}$ (Chapter 2), the Cox ring of $\overline{M}_{0,n}$ (Chapter 3), and the movable cone of $\overline{M}_{0,6}$ (Chapter 4).

We now describe our results in more detail. In Chapter 2, we address Fulton's conjecture for divisors. The conjecture is about the cone of divisor classes dual to a distinguished, finite set of effective curves called 'F-curves' (see Definition 1.2.7). Any divisor non-negatively intersecting all F-curves is called an *F-nef* divisor. Fulton's conjecture for divisors can be stated as follows:

Conjecture. *Every F-nef divisor on $\overline{M}_{0,n}$ is linearly equivalent to an effective sum of boundary divisors.*

The main motivation for studying Fulton's conjecture is that, if true, it would imply that the classes of F-curves generates the closed cone of curves, $\overline{\mathrm{NE}}(\overline{M}_{0,n})$ (this second conjecture is known as the F-conjecture). Furthermore, results of [25] show that a proof of the F-conjecture for $\overline{M}_{0,g+n}$ would imply an analogous result for $\overline{M}_{g,n}$, hence giving a relatively simple and combinatorial description of the closed cone of curves of $\overline{M}_{g,n}$, and, by duality, of its cone of nef divisors.

One advantage of studying Fulton's conjecture rather than the F-conjecture is that the former can be phrased entirely in combinatorial terms, as was realized in [25] (and is described slightly differently in Chapter 2). This apparent simplification comes at a computational cost, however. The simplest non-trivial case of $\overline{M}_{0,5}$ was checked in [25], while the cases $n \leq 6$ were subsequently proved in [18] and [21].

The main result of Chapter 2 is a proof of Fulton's conjecture for $n \leq 7$. The proof involves defining a 'canonical' subbasis of $N^1(\overline{M}_{0,n})_{\mathbb{R}}$ consisting of averages of the Keel relations among boundary divisors; we call the resulting divisor classes *Keel classes* (see Section 2.2). This subbasis is canonical in the sense that any F-nef divisor class in the span of the Keel classes can be written in a natural way as an effective sum of boundary classes. For $n \geq 6$, these classes do not span $N^1(\overline{M}_{0,n})_{\mathbb{R}}$, but for $n = 7$, we can extend the subspace generated by the Keel classes to a codimension one subspace of $N^1(\overline{M}_{0,n})_{\mathbb{R}}$ so that the same combinatorial recipe that established non-negativity of boundary coefficients for Keel classes can be used for this larger subspace of divisor classes. To finish the proof, we rewrite divisor classes outside of this codimension one subspace using particular averages of Keel relations (i.e. we move the divisor within its linear equivalence class), and then apply the simplex algorithm to check that the new boundary coefficients are all non-negative. The non-negativity can then be verified by hand (see Section 2.4). This chapter is based upon the preprint [44].

In Chapter 3 we initiate the study of relations in the Cox ring of $\overline{M}_{0,n}$ for $n \geq 6$. In the case $n = 5$, the Cox ring has been calculated in the context of del Pezzo surfaces (see [4], as well as [54], [53], and [43]). For $n = 6$, the generators of the Cox ring have been determined in [10], while for higher n, the only well-understood generators of $\mathrm{Cox}(\overline{M}_{0,n})$ arise from the boundary divisors (the existence of non-boundary generators has been established in [11], but it is not in general clear how to even calculate the numerical class of these divisors). Hence we study the ideal of relations in $\mathrm{Cox}(\overline{M}_{0,n})$ by focusing on the subring of $\mathrm{Cox}(\overline{M}_{0,n})$ generated by boundary divisor sections. To make things more precise, let $\mathcal{G}_{bnd} = \{x_J : \Delta_J \text{ a boundary divisor in } \overline{M}_{0,n}\}$ be generators of the Cox ring corresponding to boundary divisors, and let I_{bnd} be the kernel of the morphism $\mathbb{C}[\mathcal{G}_{bnd}] \to \mathrm{Cox}(\overline{M}_{0,n})$.

The first main result of Chapter 3 exhibits subrings of $\mathrm{Cox}(\overline{M}_{0,n})$ that have no non-trivial relations. A central observation in this direction is that the Cox rings of the closely-related Losev-Manin moduli spaces $\overline{L}_{n-2}$ of stable pointed chains of projective lines inject into the Cox ring of $\overline{M}_{0,n}$ (see Sections 3.2 and 3.3). Since Losev-Manin moduli spaces are toric varieties, their Cox rings are polynomial (see [14]), and can be identified with boundary divisor sections in $\mathrm{Cox}(\overline{M}_{0,n})$ as follows. If we set

$$\mathcal{G}_{bnd}(n-1,n) = \{x_J : \Delta_J \text{ a boundary divisor in } \overline{M}_{0,n}, |J \cap \{n-1,n\}| = 1\},$$

then we prove that the Cox ring of $\overline{L}_{n-2}$ is isomorphic to $\mathbb{C}[\mathcal{G}_{bnd}(n-1,n)]$, and further show that the injection into $\mathrm{Cox}(\overline{M}_{0,n})$ is, in a sense made precise below, as nice as possible.

Theorem. *There is an injection of multi-graded rings*

$$\mathbb{C}[\mathcal{G}_{bnd}(n-1,n)] \hookrightarrow \mathrm{Cox}(\overline{M}_{0,n}),$$

given by $x_J \mapsto x_J$.

In particular, this theorem shows that the ideal of relations I_{bnd} meets $\mathbb{C}[\mathcal{G}_{bnd}(n-1,n)]$ trivially, or in other words, there are no non-trivial relations among the variables $x_J \in \mathcal{G}_{bnd}(n-1,n)$. In fact, such injections exist not only for the choices $n-1$ and n, but also for any distinct $i, j \in \{1,\dots,n\}$, so we have $\binom{n}{2}$ 'relation-free' subrings. The main ingredient in the proof of this theorem is a detailed study of Kapranov's blow-up constructions of $\overline{L}_{n-2}$ and $\overline{M}_{0,n}$ (see Section 3.3). We also make use of the notion of 'clean intersections' (see Definition 3.3.8), which were first defined by Bott in [7].

In the opposite direction, the second main result of Chapter 3 can be thought of as a 'relation-full' theorem. By studying forgetful morphisms $\pi_J : \overline{M}_{0,n} \to \overline{M}_{0,n-|J|}$ (see Section 1.2.2) composed with the Kapranov morphisms $t_m : \overline{M}_{0,n-|J|} \to \mathbb{P}^{n-|J|-3}$ (see Section 1.3.3), we construct collections of divisor classes $[F_{J,m}]$ such that the $[F_{J,m}]$-graded part of $\mathrm{Cox}(\overline{M}_{0,n})$, denoted $\mathrm{Cox}(\overline{M}_{0,n})_{[F_{J,m}]}$, is a non-trivial quotient of a polynomial ring. More precisely,

Theorem. *For each* $J \subseteq \{1,\dots,n\}$, $1 \le |J| \le n-4$, *and* $m \notin J$,

$$\mathbb{C}[\mathcal{G}_{bnd}]_{[F_{J,m}]} \cap I_{bnd} \neq \emptyset,$$

and this intersection has dimension $(n-|J|-2)(n-|J|-3)/2$.

Chapter 4 is devoted to the study of the complete intersection and movable cones of curves for the three-fold $\overline{M}_{0,6}$. More than thirty years after Kleiman proved that the closure of the ample cone of a projective variety X is dual to its closed cone of curves, Boucksom, Demailly, Păun, and Peternell in [8] proved an analogous result for the cone of pseudoeffective divisors, showing that its dual cone coincides with the closure of all curves belonging to an algebraic family that covers the variety, called the *movable cone*, and denoted $\overline{\mathrm{Mov}}(X)$ (see Section 4.1 for precise definitions).

It is natural to ask if there is an alternate description of the movable cone that avoids appeals to existence of covering families of curves. The cone of *complete intersection curves*, defined on a projective variety X as the closure of the classes of curves obtained by intersecting $\dim(X)-1$ very ample divisors, and denoted $\mathcal{CI}(X)$, is one obvious candidate.

In Chapter 4, we compare the complete intersection and movable cones of the blow-ups of $\mathbb{P}^3$ that are related to Kapranov's construction of $\overline{M}_{0,6}$. Specifically, we pick r points of $\mathbb{P}^3$ in general position, $1 \leq r \leq 5$, and the $\binom{r}{2}$ lines spanned by these points, and then let X_r be the blow-up of the r points, followed by the blow-up of the proper transforms of the $\binom{r}{2}$ lines. The case $r = 4$ gives the Losev-Manin space $\overline{L}_4$, while $r = 5$ is $\overline{M}_{0,6}$. The main result of Chapter 4 is

Theorem. *For $\overline{M}_{0,6}$, there is strict containment $\mathcal{CI}(\overline{M}_{0,6}) \subsetneq \overline{\mathrm{Mov}}(\overline{M}_{0,6})$, while for $r = 1, \ldots, 4$, $\mathcal{CI}(X_r) = \overline{\mathrm{Mov}}(X_r)$.*

Noting that all varieties X_r for $1 \leq r \leq 4$ are toric, this result says that the complete intersection and movable cones diverge in the Kapranov construction after leaving the toric world. The proof of this theorem uses a combinatorial reformulation of $\mathcal{CI}(X_r)$ that makes possible an algorithmic proof, which we implement via a C++ program (see Section 4.4). We conclude the chapter with an example of a smooth projective toric three-fold for which the complete intersection cone is strictly contained in the movable cone, hence showing that even for smooth projective toric varieties, we cannot expect equality of these cones.

We now describe the organization of the remainder of this introductory chapter. In Section 1.1 we define the vector spaces and cones of divisor and curve classes that, when defined for the variety $\overline{M}_{0,n}$, are central objects of this thesis. We also establish notation and recall the projection formula, which plays a role in many of our arguments. Next, in Section 1.2, we overview the basic definitions and properties of $\overline{M}_{0,n}$ before concluding with a discussion of intersection theory on $\overline{M}_{0,n}$ in Section 1.3.

1.1 General definitions and notation

We restrict attention to smooth, irreducible, projective varieties X over $\mathbb{C}$, but the definitions below make sense in greater generality as well. We refer to [45] for further details, and to [23] and Appendix A of [31] for the basics of intersection theory.

Definition 1.1.1. Let $\mathrm{Div}_{\mathbb{R}}(X)$ be the space of $\mathbb{R}$-divisors on X. Two divisors $D_1, D_2 \in \mathrm{Div}_{\mathbb{R}}(X)$ are said to be *numerically equivalent* if for all curves $C \subseteq X$, $D_1 \cdot C = D_2 \cdot C$.

Taking the quotient by the resulting equivalence relation on $\mathrm{Div}_{\mathbb{R}}(X)$ gives the *Néron-Severi* space $N^1(X)_{\mathbb{R}}$. The dual space, denoted $N_1(X)_{\mathbb{R}}$, is the $\mathbb{R}$-vector space of one-cycles in X modulo numerical equivalence, where two one-cycles are numerically equivalent if their intersections with all effective divisors coincide. If D is a $\mathbb{R}$-divisor, we write its class in $N^1(X)_{\mathbb{R}}$ as $[D]$, and likewise elements of $N_1(X)_{\mathbb{R}}$ are written $[C]$, where C is a one-cycle on X.

Both $N^1(X)_{\mathbb{R}}$ and $N_1(X)_{\mathbb{R}}$ are finite-dimensional $\mathbb{R}$-vector spaces.

Definition 1.1.2. The *pseudoeffective cone* of divisors, written $\overline{\mathrm{Eff}}(X)$, is the closed subcone of $N^1(X)_{\mathbb{R}}$ generated by classes of effective divisors. Explicitly, $\overline{\mathrm{Eff}}(X)$ is the closure in $N^1(X)_{\mathbb{R}}$ of

$$\mathrm{Eff}(X) = \{\sum d_i[D_i] : d_i \geq 0, D_i \text{ an effective divisor on } X\}.$$

Similarly, we define the *closed cone of curves*, written $\overline{\mathrm{NE}}(X)$, as the closed subcone of $N_1(X)_{\mathbb{R}}$ generated by classes of effective curves, that is, the closure in $N_1(X)_{\mathbb{R}}$ of

$$\mathrm{NE}(X) = \{\sum c_i[C_i] : c_i \geq 0, C_i \subseteq X \text{ an irreducible curve}\}.$$

There are integral non-effective divisors that are nevertheless pseudoeffective, since a multiple of the divisor has a section (see [10], Remark 8.3, for such an example in $\overline{\mathrm{Eff}}(\overline{L}_4)$). There are also varieties with pseudoeffective divisors that cannot be written as a finite sum of effective $\mathbb{R}$-divisors, for example, the blow-up of $\mathbb{P}^2$ at nine or more general points; see [45], Section 1.5.D.

Definition 1.1.3. The cone of *nef divisors* in X is

$$\mathrm{Nef}(X) = \{[D] \in N^1(X)_{\mathbb{R}} : D \cdot C \geq 0 \text{ for all } [C] \in \overline{\mathrm{NE}}(X)\} = (\overline{\mathrm{NE}}(X))^{\vee}.$$

We will often reduce intersection properties on $\overline{M}_{0,n}$ to intersections on a more amenable variety via the so-called projection formula. We will use this result in the case of Chow rings on non-singular varieties, where the formula takes on a particularly simple form (see [23], Proposition 8 (c)).

Lemma 1.1.4 (Projection formula)**.** *Let $f : X \to Y$ be a proper morphism of non-singular varieties. For $x \in A^*(X)$ and $y \in A^*(Y)$,*

$$f_*(x \cdot f^*y) = f_*(x) \cdot y.$$

Notation 1.1.5. On many occasions we will designate a cone, projective subspace, or vector subspace cone by generating elements. For a cone in a real vector space V generated by a finite subset $\mathcal{G} \subseteq V$, we write

$$\langle g : g \in \mathcal{G}\rangle_{\geq 0} = \{\sum_{g \in \mathcal{G}} \lambda_g g : \lambda_g \geq 0\}.$$

For a real vector subspace of V generated by a finite subset of vectors $\mathcal{G}$, we write the span of $\mathcal{G}$ as

$$\langle g : g \in \mathcal{G}\rangle = \{\sum_{g \in \mathcal{G}} \lambda_g g : \lambda_g \in \mathbb{R}\},$$

while for a projective subspace generated by a finite number of points indexed by $\mathcal{G} \subseteq \mathbb{P}^N$, we write

$$\langle g : g \in \mathcal{G}\rangle = \text{the smallest projective subspace containing all } g \in \mathcal{G}.$$

1.2 Basics of $\overline{M}_{0,n}$

Our main sources for the remainder of this chapter are [1], [38], [41], [29], and [42], the first two chapters of which give perhaps the gentlest introduction to the ideas presented here. We quickly introduce the most basic notions, and then describe intersection theory on $\overline{M}_{0,n}$ in greater detail in the next section. We define $\overline{M}_{0,n}$ over $\mathbb{C}$ although it can be defined more generally over any algebraically closed field.

Definition 1.2.1. A *stable n-pointed rational curve* is a curve C along with distinct points $p_1, \dots, p_n \in C$ satisfying the following properties:

(i) C is connected and reduced, with each irreducible component isomorphic to $\mathbb{P}^1$;

(ii) the intersecting components of C meet at ordinary double points;

(iii) $H^1(C, \mathcal{O}_C) = 0$;

(iv) p_i is a smooth point of C for all i; and

(v) each irreducible component of C contains at least three *special points*, where a special point is a node or one of the p_i.

We write such a curve as $(C, p_1, \dots, p_n)$.

Conditions (i)–(iii) imply that C is a tree of projective lines, while condition (v) ensures stability, i.e. that the automorphism group of C is trivial.

Stable n-pointed curves are defined in families as follows:

Definition 1.2.2. A *family of stable n-pointed rational curves* over a scheme S is a flat proper morphism $\pi : F \to S$ having n sections, $\sigma_1, \dots, \sigma_n$, such that each geometric fiber $(F_s, \sigma_1(s), \dots, \sigma_n(s))$ of π is a stable n-pointed rational curve. We denote such a family of curves by $(\pi : F \to S, \sigma_1, \dots, \sigma_n)$.

Definition 1.2.3. Two families of stable n-pointed curves $(\pi : F \to S, \sigma_1, \dots, \sigma_n)$ and $(\pi' : F' \to S, \sigma'_1, \dots, \sigma'_n)$ are *isomorphic* if there exists an isomorphism $f : F \to F'$ over S such that $f \circ \sigma_i = \sigma'_i$ for all i.

To define a moduli space, for each scheme S we let $\mathbf{F}(S)$ be the set of all families of stable n-pointed curves over S modulo the equivalence relation $\sim$ determined by isomorphisms. Then $\mathbf{F}$ is a (contravariant) functor from schemes over $k = \mathbb{C}$ to sets. In [41], Knudsen proved that the functor $\mathbf{F}$ is representable by a smooth projective variety $\overline{M}_{0,n}$ of dimension $n-3$. In particular, Knudsen showed that the universal curve over $\overline{M}_{0,n}$ is a copy of $\overline{M}_{0,n+1}$ mapped to $\overline{M}_{0,n}$ by the forgetful morphism forgetting the $(n+1)^{\text{st}}$ point (see Section 1.2.2 for a definition of forgetful morphisms).

The variety $\overline{M}_{0,n}$ is a modular compactification of the moduli space of n ordered points on the projective line modulo projective equivalence, denoted $M_{0,n}$, with the points of the boundary $\overline{M}_{0,n} \setminus M_{0,n}$ precisely the singular stable curves defined above.

Example 1.2.4. In Figure 1.1 are depicted two points on the boundary of $\overline{M}_{0,5}$. Since the general linear group acting on $\mathbb{P}^1$ is three-transitive, on a given irreducible component, we may take any three special points to be 0, 1, and ∞. Hence replacing the vertically drawn component in Figure 1.1 (a) with any other rational component containing p_1, p_2, and the node gives the same element of $\overline{M}_{0,5}$. For the horizontal component, if we take the node, p_3, and p_4 to be 0, 1, and ∞, respectively, then the coordinate of p_5 uniquely determines the isomorphism class.

Figure 1.2 likewise shows two points on the boundary of $\overline{M}_{0,6}$.

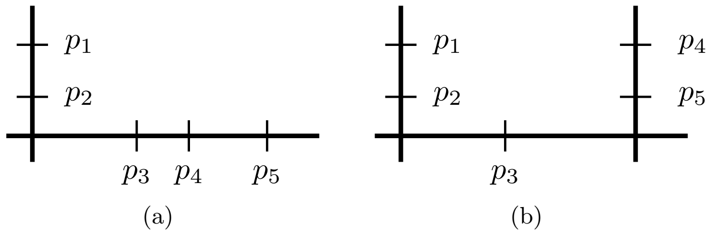


Figure 1.1: Elements of $\overline{M}_{0,5}$

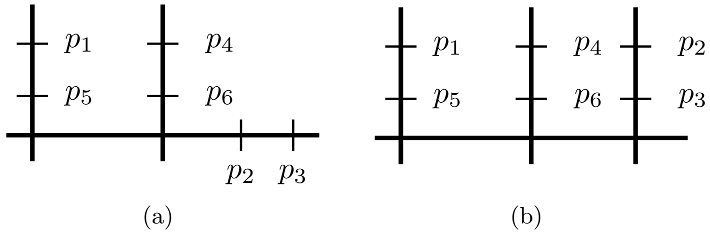


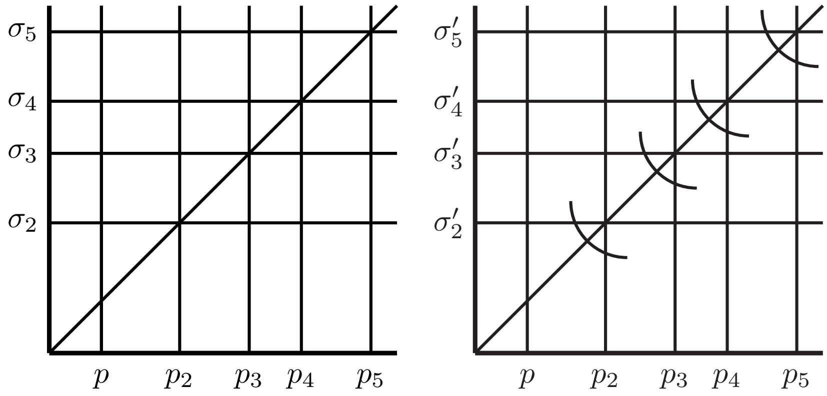
Figure 1.2: Elements of $\overline{M}_{0,6}$

1.2.1 Stabilization

The Mumford-Knudsen compactification describes what happens in the limit as two distinct marked points on $\mathbb{P}^1$ approach one another. Informally, the limit of the points p_i and p_j coming together is obtained by adding a new rational component containing p_i and p_j. This intuition can be made precise via Definition 1.2.2. We consider first an example in $\overline{M}_{0,5}$. Let $S = \mathbb{P}^1$, and to start with let $F = \mathbb{P}^1 \times \mathbb{P}^1$ and let $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection onto the first factor with sections $\sigma_1(p) = (p, p)$, and $\sigma_i(p) = (p, p_i)$ for $i = 2, \dots, 5$ (see Figure 1.3 (a)). When $p \neq p_i$, the fiber F_p is a rational curve with five distinct marked points (i.e. an element of $M_{0,5}$), with $p_1 = \sigma_1(p)$ representing a moving point. The fibers above the p_i, $i = 2, \dots, 5$, are not, however, stable n-pointed curves.

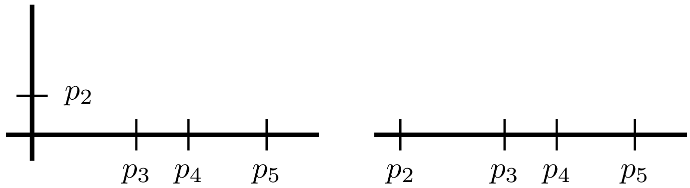
The remedy of Mumford-Knudsen is to blow-up the points of intersection of sections and the diagonal. This process is called *stabilization.* Let F' be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points (p_2, p_2), (p_3, p_3), (p_4, p_4), and (p_5, p_5), and label the proper-transforms of the sections by $\sigma_2', \dots, \sigma_5'$. If $\pi' : F' \to \mathbb{P}^1$ is the composition of the blow-down map and projection, then the fibers F'_{p_i} satisfy criteria (i) - (iv) of Definition 1.2.1; see Figure 1.3 (b). For example, the fiber over p_2 is shown in Figure 1.1 (a). It should be noted that this is not the only compactification of $M_{0,n}$. We will make use of another compactification due to Losev-Manin in Chapters 3 and 4, and these two belong to a spectrum of compactifications whose study was initiated by Hassett in [32].

There is nothing special about beginning with an element of $M_{0,5}$ in the above procedure. The same construction carries over to a moving point of any component of a stable pointed curve. For example, if in the curve of Figure 1.1 (a) we let the point p_5



(a) (b)

Figure 1.3: Family of curves in $\overline{M}_{0,5}$



(a) An unstable image of π'_1 (b) The contraction of the curve in (a)

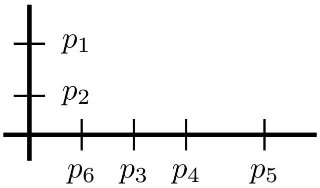
Figure 1.4: Example of contraction in $\overline{M}_{0,4}$

approach p_4, the limit is the curve shown in Figure 1.1 (b). In particular, if we take the closure of the irreducible locus of points in $\overline{M}_{0,5}$ having marked points p_1 and p_2 on one component, and p_3, p_4, and p_5 on the second component, then the curve of (b) is contained in this subvariety. The same holds for the elements of $\overline{M}_{0,6}$ shown in Figure 1.2: the curve depicted in (b) is contained in the closure of points whose general element is as in (a). These examples are simple cases of a stratification of $\overline{M}_{0,n}$.

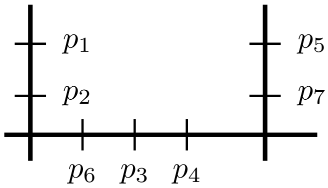
1.2.2 Contraction and forgetful morphisms

Besides stabilization, a second essential operation on $\overline{M}_{0,n}$ is *contraction*. We consider first an example involving $\overline{M}_{0,5}$. Let π'_1 be the map from $\overline{M}_{0,5}$ that forgets the marked point p_1. For example, the image of the curve depicted in Figure 1.1(a) is the curve in Figure 1.4 (a). This curve is not stable, since there are only two special points on the component containing the marked point p_2. To obtain an element of $\overline{M}_{0,4}$, we contract this rational component, as shown in Figure 1.4 (b). We denote the resulting map $\pi_1 : \overline{M}_{0,5} \to \overline{M}_{0,4}$, and call π_i the *forgetful morphism* forgetting the marked point p_i.

In general, let E be the irreducible component of an element of $\overline{M}_{0,n}$ containing the



(a) An element of $\overline{M}_{0,6}$



(b) Image under σ_5

Figure 1.5: Example of $\sigma_5 : \overline{M}_{0,6} \to \overline{M}_{0,7}$

marked point p_i. If E contains in addition to p_i at least three other special points, then the maps π_i' and π_i defined above coincide. Otherwise, π_i is defined by π_i' followed by contracting E. Knudsen showed in [41] that contraction carries over to a morphism of families of pointed rational curves, giving also forgetful morphisms of families, and that these operations commute with fibered products.

Knudsen in [41] further identifies the forgetful morphism $\pi_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ with the universal curve over $\overline{M}_{0,n}$, with sections $\sigma_1, \ldots, \sigma_n$ defined as follows. The image of $(C, p_1, \ldots, p_n) \in \overline{M}_{0,n}$ under $\sigma_i : \overline{M}_{0,n} \to \overline{M}_{0,n+1}$ is obtained by gluing to the point $p_i \in C$ a copy of $\mathbb{P}^1$ by identifying $0 \in \mathbb{P}^1$ with $p_i \in C$, and labeling the points $1, \infty \in \mathbb{P}^1$ as p_i, p_{n+1}, respectively. Note that, up to automorphism, it does not matter where the points p_i and p_{n+1} are placed on the new component so long as they are distinct from one another and the singular point, since the automorphism group of the projective line is three-transitive (see Figure 1.5 for an example of $\sigma_5 : \overline{M}_{0,6} \to \overline{M}_{0,7}$).

1.2.3 Stratification of $\overline{M}_{0,n}$

The stratification of $\overline{M}_{0,n}$ by topological type is as follows: a *codimension one-stratum* is an irreducible component of the locus of points of $\overline{M}_{0,n}$ having at least one node (equivalently, at least two components). Hence the generic element of a codimension one-stratum has two components, with some $J \subseteq \{1, \ldots, n\}$, $2 \leq |J| \leq n-2$, giving the marked points on one component, and J^c giving the marked points on the other.

The *codimension 2-strata* are likewise the irreducible components of loci of points of $\overline{M}_{0,n}$ having at least two nodes. We continue until reaching the *dimension one-strata*, which are irreducible components of loci with at least $n-4$ nodes (or at least $n-3$ components).

Definition 1.2.5. A codimension one-stratum is called a *boundary divisor*, and is denoted by Δ_J, where $J \subseteq \{1, \ldots, n\}$, $2 \leq |J| \leq n-2$, records the marked points on one component of the general element, while J^c label the marked points on the other component.

This stratification has the following nice properties (see [39], p. 8):

(i) The union of the boundary divisors (with reduced structure) is a normal crossing divisor.

(ii) Each codimension k-stratum is uniquely (up to ordering) a complete intersection of k boundary divisors.

There are multiple conventions for how to uniquely label each boundary divisor in $\overline{M}_{0,n}$. The usual choice is to stipulate that $|J| \leq n/2$, and if $|J| = n/2$, then $1 \in J$. Several of our results function best with labeling schemes different from this usual one (see, for example, Proposition 2.2.7 and Section 3.3), while other results are completely independent of the scheme chosen. We will only specify a labeling scheme when one facilitates discussion or a calculation.

Notation 1.2.6. Any J used with a boundary divisor Δ_J is assumed to satisfy $J \subseteq \{1, \ldots, n\}$, $2 \leq |J| \leq n-2$, and some labeling scheme to ensure that $\Delta_J = \Delta_{J'}$ if and only if $J = J'$.

In addition to boundary divisors, attention will be mostly focused on the dimension one-strata, and their numerical properties.

Definition 1.2.7. Any curve in $\overline{M}_{0,n}$ numerically equivalent to a dimension 1-stratum is called a *Faber* or *F-curve*.

Let F be a 1-stratum, and define $G = G(F)$ to be its generic member.

Lemma 1.2.8. *$G = G(F) \in \overline{M}_{0,n}$ has $n-3$ irreducible components, with precisely one component containing four special points, and the others containing three special points.*

Proof. Let $G_1, \ldots, G_{n-3}$ be the irreducible components of G. If s_i is the number of special points on the component G_i, then $s_1 + \ldots + s_{n-3} = n + 2(n-4)$. Suppose that s_1 is the largest of the s_i. Then by the stability condition (v) of Definition 1.2.1, it follows that $3n - 8 - s_1 \geq 3(n-4)$, or $s_1 \leq 4$, but if $s_1 = 3$, there would be $3n-9$ special points, a contradiction. □

Now let $Q = Q(F)$ be the component with four special points. Then $G \setminus Q$ has as many connected components as there are singular points on Q.

Definition 1.2.9. Let F be a one-stratum. The component of $G(F)$ containing four special points is called the *spine* of F, and connected components of $G \setminus Q$ are called the *tails* of F.

1.3 Intersection theory on $\overline{M}_{0,n}$

In this section, we describe three ways to compute intersections of curves and divisors in $\overline{M}_{0,n}$. The first method is from [39] and uses combinatorics of F-curves, the second uses Keel's computation of the Chow ring of $\overline{M}_{0,n}$ from [38], and the third is via Kapranov's blow-up construction of $\overline{M}_{0,n}$ from [36] and [37].

1.3.1 Intersections via combinatorics of F-curves

Since the boundary divisors generate $N^1(\overline{M}_{0,n})_{\mathbb{R}}$ as a vector space, all intersections of divisors with F-curves are determined once we know how F-curves intersect with boundary divisors. By the definition of an F-curve, we may restrict our attention to intersections with one-strata in $\overline{M}_{0,n}$. As noted in the previous section, the general member G of a one-stratum F has a unique component, called the spine of F, with four special points. Since G is a tree of projective lines, if we label the special points of the spine by a, b, c, and d, the special points determine a partition of $\{1, \dots, n\}$ into four subsets $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, with $\mathcal{A}$ being the marked points on the union of components of G that attach to the spine at the special point a, and likewise for $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. For example, if we take the element of $\overline{M}_{0,5}$ depicted in Figure 1.1 (a) as the general member of a one-stratum, then the corresponding partition is $(\{1,2\},\{3\},\{4\},\{5\})$. Taking instead the element of $\overline{M}_{0,6}$ from Figure 1.2 (a) as the general member of a one-stratum, the partition is $(\{1,5\},\{4,6\},\{2\},\{3\})$.

In [39] it is proved that the partition defined by a one-stratum F uniquely determines the numerical equivalence class of F by showing that the intersection of a one-stratum with an arbitrary boundary divisor is determined by the partition.

Proposition 1.3.1. *Let F be a one-stratum of $\overline{M}_{0,n}$, and let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ be the corresponding partition of $\{1, \dots, n\}$. For any boundary divisor Δ_J,*

$$F \cdot \Delta_J = \begin{cases} -1 & \textit{if } J \textit{ or } J^c \textit{ equals one of the partitions,} \\ 1 & \textit{if the union of two partitions equals } J, \\ 0 & \textit{otherwise.} \end{cases}$$

Note that these formulae are valid regardless of which labeling scheme we use to label boundary (see Notation 1.2.6), since each of the conditions is symmetric in J and J^c.

For the proof, Keel and McKernan use the combinatorics of the one-strata to reduce the intersection of F with Δ_J to the self-intersection of F in a two-stratum, which is isomorphic to either $\overline{M}_{0,5}$ or $\overline{M}_{0,4} \times \overline{M}_{0,4}$. We refer to [39], Lemma 4.3 for further details.

Corollary 1.3.2. *Up to numerical equivalence, there are a finite number of F-curves.*

Proof. The number of F-curves, modulo numerical equivalence, is precisely the number of different ways to partition $\{1, \dots, n\}$ into four subsets. □

By definition, then, we can specify any F-curve by the partition defined by any one-stratum numerically equivalent to it.

Notation 1.3.3. For an F-curve with corresponding partition $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, we denote the class of the F-curve by $F(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$.

To reduce clutter, subsets of $\{1, \dots, n\}$ are written without brackets, and sets with more than one element are stacked vertically; for example, $F(1, 2, 3, {}^4_5)$ is an F-curve in $\overline{M}_{0,5}$.

It will be convenient to group F-curves by the shape of the corresponding partition. We will call this the *type* of the F-curve.

Definition 1.3.4. Let $F = F(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ be an F-curve in $\overline{M}_{0,n}$. The *type* of F is defined as the four-tuple of partition element cardinalities, $\mathbf{t} = \{|\mathcal{A}| : |\mathcal{B}| : |\mathcal{C}| : |\mathcal{D}|\} \in \mathrm{Sym}(\mathbb{N}^4)$.

Note that the type of F is defined modulo permutations of the four-elements of the partition, but this ambiguity is often preferable to fixing an ordering of the cardinalities of the elements of the partition.

1.3.2 Intersections via Keel's presentation of the Chow ring

If we limit ourselves to the strata of $\overline{M}_{0,n}$, a second way to compute intersections is to use the fact that all strata are complete intersections of boundary divisors along with Keel's presentation of the Chow ring of $\overline{M}_{0,n}$, which we now describe.

In [38], Keel proved that the Chow ring $A^*(\overline{M}_{0,n})$ isomorphic to the polynomial ring on symbols Δ_J corresponding to boundary divisors, modulo the following relations:

(i) $\Delta_J = \Delta_J^c$;

(ii) for distinct elements $i, j, k, l \in \{1, \ldots, n\}$,

$$\sum_{\substack{\{i,j\}\subseteq J,\\ \{k,l\}\subseteq J^c}} \Delta_J = \sum_{\substack{\{i,k\}\subseteq J,\\ \{j,l\}\subseteq J^c}} \Delta_J = \sum_{\substack{\{i,l\}\subseteq J,\\ \{j,k\}\subseteq J^c}} \Delta_J; \tag{1.3.1}$$

(iii) $\Delta_{J_1} \cdot \Delta_{J_2} = 0$ unless one of the following is true:

$$J_1 \subseteq J_2, \quad J_2 \subseteq J_1, \quad J_1 \subseteq J_2^c, \quad J_1^c \subseteq J_2. \tag{1.3.2}$$

Now let S be a codimension k-stratum, and let T be a codimension l-stratum of $\overline{M}_{0,n}$. By Property (ii) of the stratification in Section 1.2.3, there are unique boundary divisors Δ_{J_s} and Δ_{J_t} such that

$$S = \Delta_{J_{s_1}} \cdot \ldots \cdot \Delta_{J_{s_k}}, \text{ and } T = \Delta_{J_{t_1}} \cdot \ldots \cdot \Delta_{J_{t_l}}.$$

Assuming that $k + l \leq n$, if all boundary divisors are distinct, then $S \cdot T = 0$ if any pair of the divisors does not satisfy one of the relations as in (1.3.2) above. If each pair of divisors satisfies one of these relations, and all boundary divisors are distinct, then $S \cdot T$ is the transverse intersection of the boundary divisors occuring in S and T.

It remains to consider the case of a non-zero intersection in which at least one boundary divisor appears at least twice. Suppose for example that Δ_J appears twice in the expressions for S and T. Then we can use the relations in Equation (1.3.1) to replace one of the expressions Δ_J with a sum of boundary divisors, none of which is Δ_J, thus giving us a transverse intersection. This process can be iterated for further matching boundary divisors, and must terminate, or else the Chow ring of $\overline{M}_{0,n}$ would not be completely determined by Keel's presentation.

Example 1.3.5. To calculate the self-intersection $[\Delta_{126}] \cdot [\Delta_{126}] \in N_1(\overline{M}_{0,6})$, we move one of the $[\Delta_{126}]$ within its numerical equivalence class via the Keel relation given by the four-tuple (1234). We take the first equality of Equation (1.3.1), which we can represent as (12)(34) = (13)(24). The result is

$$\begin{aligned}[\Delta_{126}] \cdot [\Delta_{126}] &= [\Delta_{126}] \cdot ([\Delta_{13}] + [\Delta_{135}] + [\Delta_{136}] + [\Delta_{24}] - [\Delta_{12}] - [\Delta_{125}] - [\Delta_{34}]) \\ &= -[\Delta_{126}] \cdot ([\Delta_{12}] + [\Delta_{34}]) \\ &= -F(3,4,5,\substack{1\\2\\6}) - F(1,2,6,\substack{3\\4\\5}).\end{aligned}$$

Note that this expression is not unique due to the choice of Keel relation.

1.3.3 Kapranov blow-up construction

To finish this introductory chapter, we describe Kapranov's blow-up construction of $\overline{M}_{0,n}$, which threads though each of the subsequent chapters. Besides providing a convenient choice of basis for $N^1(\overline{M}_{0,n})_{\mathbb{R}}$ (and dually for $N_1(\overline{M}_{0,n})_{\mathbb{R}}$), Kapranov's construction also provides a straightforward method of intersecting psi-classes (defined below) with F-curves.

Kapranov's construction begins with the classical result of Castelnuovo that there exists a unique rational normal (or Veronese) curve through any $n+1$ points of $\mathbb{P}^{n-2}$ in general linear position. Alternatively, if we fix n points in general linear position plus a tangent direction at one of the points x_i, then there likewise exists a unique Veronese curve in $\mathbb{P}^{n-2}$ passing through the n points with the specified tangent space at x_i.

The connection to $\overline{M}_{0,n}$ is already clear: each Veronese curve passing through the points $x_1, \ldots, x_n$ defines a rational curve with n-marked points, and hence gives an element of $M_{0,n}$. To extend the correspondence to stable curves, if $(C, p_1, \ldots, p_n) \in \overline{M}_{0,n}$, it is proved in [41] that the twisted dualizing sheaf $\Omega_C(p_1 + \ldots p_n)$ is a very-ample line bundle with $n-1$ linearly independent global sections. Letting $x_i \in \mathbb{P}^{n-2}$ be the image of p_i under the embedding induced by $\Omega_C(p_1 + \ldots + p_n)$, for C smooth, the image of C is a Veronese curve through $x_1, \ldots, x_n$. For a reducible curve $C = C_1 \cup \ldots \cup C_r$ with components $C_i \cong \mathbb{P}^1$ and special points $q_1, \ldots, q_{t_i} \in C_i$, Kapranov proved that the component C_i is sent to a Veronese curve in the projective span of its image that passes through the images of the special points (see Theorem 2.3 (a) of [37]). The central result of [37] is that $\overline{M}_{0,n}$ is isomorphic to the closure in the Hilbert and Chow schemes of the subvariety of Veronese curves through fixed general points $x_1, \ldots, x_n \in \mathbb{P}^{n-2}$. For the remainder of this section, we will distinguish between an element of $\overline{M}_{0,n}$ and its realization as a Veronese curve (or union of Veronese curves) only by the labels of the marked points: elements of $\overline{M}_{0,n}$ will be labeled as $(C, p_1, \ldots, p_n)$, while the image of this curve in $\mathbb{P}^{n-2}$ will be denoted $(C, x_1, \ldots, x_n)$.

To connect the realization of $\overline{M}_{0,n}$ via Veronese curves through $x_1, \ldots, x_n \in \mathbb{P}^{n-2}$ to a blow-up of $\mathbb{P}^{n-3}$, we choose the tangent direction at the point $x_n \in \mathbb{P}^{n-2}$. Consider the morphism $t_n : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ that sends a curve C to its embedded tangent line at x_n; we call the n^{th} marked point in this case the *moving point*. Then the fiber of $t_n^*(\mathcal{O}_{\mathbb{P}^{n-3}}(1))$ at a curve $(C, x_1, \ldots, x_n)$ is the cotangent space of C at the marked point x_n, which is the fiber of the the psi-class ψ_n at C, defined as follows:

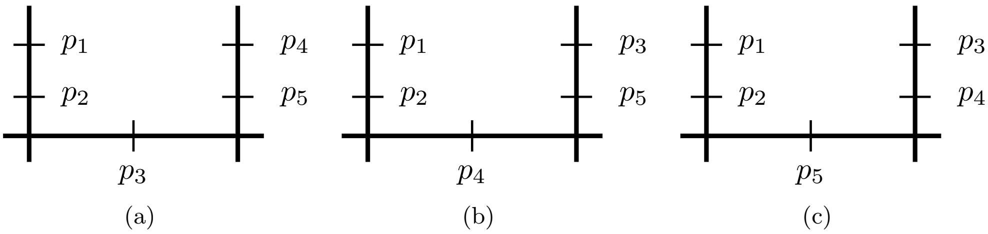


Figure 1.6: Elements of $\Delta_{12} \subseteq \overline{M}_{0,5}$ corresponding to Veronese curves

Definition 1.3.6. Let $\pi_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ be the universal curve over $\overline{M}_{0,n}$ with sections $\sigma_1, \ldots, \sigma_n$ as described in Section 1.2.2, and let $\omega_{\pi_{n+1}}$ be the relative dualizing sheaf. Then for $i = 1, \ldots, n$, the *psi-class* ψ_i is

$$\psi_i = \sigma_i^*(c_1(\omega_{\pi_{n+1}})).$$

Over a stable n-pointed curve, $(C, p_1, \ldots, p_n) \in \overline{M}_{0,n}$, the fiber of ψ_i is $T^*_{p_i}C$, the contangent space to C at p_i.

Example 1.3.7. Fix five general points $x_1, \ldots, x_5 \in \mathbb{P}^3$. The elements of $M_{0,5}$ are twisted cubics through the points $x_1, \ldots, x_5$. To understand the boundary $\overline{M}_{0,5} \setminus M_{0,5}$, consider for example the element $(C, p_1, \ldots, p_5)$ of Figure 1.1 (a), which has two components, C_1 and C_2, with special points p_1, p_2, q' on C_1 and q', p_3, p_4, p_5 on C_2, where q' is the point of attachment of C_1 and C_2. Since each irreducible component is sent to a Veronese curve in its projective span, the image of C_1 is a projective line containing the points x_1 and x_2, while the image of C_2 is a plane conic containing the points x_3, x_4, and x_5; the line and conic meet at the image of the point q', which we denote by $q \in \mathbb{P}^3$.

Next we consider the image under $t_5 : \overline{M}_{0,5} \to \mathbb{P}^2$ of the boundary divisor Δ_{12} containing $(C, p_1, \ldots, p_5)$ as a general element. The elements of Δ_{12} correspond to the pencil of conics through the points q, x_3, x_4, and x_5, so the embedded tangent lines to $C \in \Delta_{12}$ at x_5 are sent by t_5 to a line in $\mathbb{P}^2$. In particular, the three singular points of the pencil are the pairs of intersecting lines $\langle x_4, x_5\rangle \cup \langle q, x_3\rangle$, $\langle x_3, x_5\rangle \cup \langle q, x_4\rangle$, and $\langle q, x_5\rangle \cup \langle x_3, x_4\rangle$. The corresponding three elements of $\overline{M}_{0,5}$ are shown, respectively, in Figure 1.6.

It is easy to see which line in $\mathbb{P}^2$ is the image of Δ_{12} by considering the images corresponding to the three singular points of the pencil. For $i = 1, \ldots, 4$, define $y_i \in \mathbb{P}^2$ be the image under t_5 of the line $\langle x_i, x_5\rangle \subseteq \mathbb{P}^3$. With this notation, the element shown in (a) is mapped to $y_4 \in \mathbb{P}^2$, the element of (b) is mapped to $y_3 \in \mathbb{P}^2$, and the element of (c) is mapped to the intersection of the line $\langle y_3, y_4\rangle$ with the line at infinity, and so the image of Δ_{12} is the line $\langle y_3, y_4\rangle \subseteq \mathbb{P}^2$.

If instead we consider the image of a boundary divisor Δ_{i5}, $i = 1, \ldots, 4$, the situation is even simpler. Every element of Δ_{i5} can be written $C = C_1 \cup C_2$, with the marked points p_i and p_5 on the component C_1, and the marked points in the complement on the (possibly reducible) component C_2. In this case, the embedded tangent space to all such C at x_5 is image is the line $\langle x_i, x_5\rangle$, and so the image of Δ_{i5} under t_5 is just the point $y_i \in \mathbb{P}^2$.

The above example already contains the main ingredients for intersecting F-curves with psi-classes.

Proposition 1.3.8. *Let F be a one-stratum of $\overline{M}_{0,n}$, and let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ the the corresponding partition of $\{1, \ldots, n\}$. Then for each $i = 1, \ldots, n$,*

$$\psi_i \cdot F = \begin{cases} 1 & \textit{if one of the partitions equals } \{p_i\}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $G = G(F)$ be the general member of the one-stratum F. Then one of the partitions determined by F is equal to the singleton set $\{p_i\}$ precisely when p_i is on the spine of F, $Q = Q(F)$ (see Definition 1.2.9). By Lemma 1.2.8, the spine of F is the unique irreducible component of G with four special points, with all other irreducible components having three special points. Hence the Veronese realization of the irreducible component of G containing p_i is a plane conic if p_i is on the spine, and a projective line otherwise, and the image of the component of the one-stratum F containing p_i is either a pencil of conics in the former case, or a projective line in the latter. By the projection formula applied to the morphism $t_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$,

$$\psi_i \cdot F = t_i^*(\mathcal{O}_{\mathbb{P}^{n-3}}(1)) \cdot F = \mathcal{O}_{\mathbb{P}^{n-3}}(1) \cdot t_{i*}(F),$$

but $t_{i*}(F)$ is a projective line (and hence $\psi_i \cdot F = 1$) precisely when a component of the Veronese realization of F is a pencil of conics containing x_i; otherwise the component of the Veronese realization of F containing x_i is a fixed projective line, hence in this case $t_{i*}(F) = 0$, and so $\psi_i \cdot F = 0$. □

It is already evident from the example of $\overline{M}_{0,5}$ that the morphism $t_5 : \overline{M}_{0,5} \to \mathbb{P}^2$ can be resolved by blow-ups involving the points $y_1, \ldots, y_4 \in \mathbb{P}^2$. In [36], Kapranov described for all n an explicit series of blow-ups through which the morphism $t_n : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ factors, and proved that $\overline{M}_{0,n}$ is isomorphic to the resulting blow-up of $\mathbb{P}^{n-3}$. The blow-ups can be performed in the following order: first we blow-up the points $y_1, \ldots, y_{n-1}$, then the proper transforms of the lines $\langle y_1, y_2 \rangle, \ldots, \langle y_{n-2}, y_{n-1} \rangle$, and continue blowing up the proper transforms of the linear subspaces spanned by the points until all codimension two subspaces have been blown up.

There are subtleties involved in the ordering of these blow-ups. The order given here is due to Hassett (see [32]); Kapranov's original ordering of the blow-ups, and the relation among different orderings, is a focus of Chapter 3.

The blow-up construction also gives a basis for $N^1(\overline{M}_{0,n})_\mathbb{R}$. Let H be the pull-back to $\overline{M}_{0,n}$ of a general hyperplane in $\mathbb{P}^{n-3}$ under the composition of the blow-down morphisms, and for $J \subseteq \{1, \ldots, n-1\}$, $1 \le |J| \le n-4$, let E_J be the proper transform of the exceptional divisor obtained from blowing up the linear subspace $\langle y_j : j \in J \rangle$.

Definition 1.3.9. The *Kapranov basis* of $\overline{M}_{0,n}$ is

$$\{[H], [E_J] : J \subseteq \{1, \ldots, n-1\}, 1 \le |J| \le n-4\} \subseteq N^1(\overline{M}_{0,n})_\mathbb{R}.$$

Kapranov proved in [37] that the hyperplane class equals the psi-class of the moving point, so taking the n^{th} point as the moving point, $[H] = \psi_n$ (this equality is also mentioned before Definition 1.3.6 using the fact that the Veronese realization $t_n : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ can be factored as a composition of blow-ups). In addition, the correspondence between Kapranov's realization of $\overline{M}_{0,n}$ via Veronese curves and the blow-up construction yields a dictionary between the exceptional divisors arising the the blow-up construction and the boundary divisors:

$$\begin{aligned} \Delta_{J\cup\{n\}} &= E_J, \text{ if } 1 \le |J| \le n-4, \\ [\Delta_{J\cup\{n\}}] &= [H] - \Big(\sum_{J' \subsetneq J} [E_{J'}] \Big), \text{ if } |J| = n-3. \end{aligned} \tag{1.3.3}$$

If $\{a, b\} = (J \cup \{n\})^c$ for $|J| = n-3$, the second set of equalities can be rewritten as

$$[\Delta_{ab}] = [H] - \Big(\sum_{J' \subsetneq \{a,b,n\}^c} [E_{J'}] \Big). \tag{1.3.4}$$

We will prove in Proposition 3.3.12 that, up to isomorphism, the dictionary is the same for the Hassett or Kapranov orderings of the blow-ups.

2 Fulton's conjecture for $\overline{M}_{0,7}$

2.1 Introduction

A central open problem in the birational geometry of the moduli space of stable pointed rational curves, $\overline{M}_{0,n}$, is the F-conjecture, which posits that the closed cone of curves of $\overline{M}_{0,n}$ is generated by F-curves (see Definition 1.2.7). This conjecture has been proven for $n \leq 7$ in [39] using techniques from the minimal model program and negativity properties of the canonical bundle that do not hold for higher n.

It was realized in [25] that the F-conjecture is implied by another conjecture that can be stated in terms of convex geometry of finite dimensional vector spaces. This conjecture, which we call Fulton's conjecture for divisors, first appeared in [39], and was proven for $n \leq 6$ in [18], and, independently and by different methods, in [21]. The main result of this paper is a proof of Fulton's conjecture for $n = 7$. We begin by describing the usual formulation of the conjecture.

The stratification of $\overline{M}_{0,n}$ described in Section 1.2.3 led Fulton to ask if the effective k-cycles of $\overline{M}_{0,n}$ were generated by the k-strata, as is the case with toric varieties. The question has a negative answer for divisors (see [55]), but by restricting to divisors non-negatively intersecting all F-curves, called *F-nef divisors*, we obtain

Conjecture 2.1.1 (Fulton's conjecture)**.** Every F-nef divisor is numerically equivalent to an effective sum of boundary divisors.

To interpret Fulton's conjecture in terms of cones in finite-dimensional vector spaces, let V be the $2^{n-1} - n - 1$ dimensional vector space over $\mathbb{Q}$ with standard basis elements labeled Δ_{J,J^c}, where $J \subseteq \{1, \ldots, n\}$, and $|J|, |J^c| \geq 2$. Since the numerical equivalence classes of boundary divisors generate the Néron-Severi space, $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$, there is a surjection $\phi : V \to N^1(\overline{M}_{0,n})_{\mathbb{Q}}$ with kernel $\mathcal{I}$. The subspace $\mathcal{I}$ is generated by the Keel relations among boundary divisors (see Equation (1.3.1)). Lastly, let $\mathcal{F} \subseteq N^1(\overline{M}_{0,n})_{\mathbb{Q}}$ be the cone of F-nef divisor classes. We can thus restate the conjecture as follows:

Conjecture 2.1.2 (Fulton's conjecture, convex geometry formulation)**.** For every $\alpha \in \mathcal{F}$, $\phi^{-1}(\alpha)$ intersects the positive orthant $\langle \Delta_{J,J^c} \rangle_{\geq 0}$ nontrivially.

By an inductive argument which we now describe, Fulton's conjecture implies that the nef cone and the cone of F-nef divisors coincide (see also [49]); it is not clear that implication holds in the other direction. Suppose D is F-nef and can be written as an effective sum of boundary divisors. For an irreducible curve C and an effective divisor E, if C is not contained in the support of E, then $C \cdot E \geq 0$. Hence we need only consider curves $C \subseteq \Delta_J$, where Δ_J is a boundary divisor in the support of D. Every boundary divisor Δ_J is isomorphic to a product $\overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1}$, where $n_1+n_2 = n$, and n_1 and n_2 are at least 2 (see [39]). This isomorphism is obtained by gluing the $(n_i + 1)^{st}$ points of each factor. But the pullback of D under the composition of the isomorphism $\overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1} \to \Delta_J$ and the inclusion $\Delta_J \hookrightarrow \overline{M}_{0,n}$ gives F-divisors D_1 and D_2 on each of the factors. Since both D_1 and D_2 are effective sums of boundary divisors on $\overline{M}_{n_1}$, $\overline{M}_{n_2}$, respectively, by induction, the pull-back of C intersects D_1 and D_2 non-negatively, and so $C \cdot D \geq 0$, showing that D is nef. In particular, the nef cone equals the cone of F-divisors, which by duality implies the F-conjecture.

Either conjecture, if true, would yield surprising consequences. It would follow that, on the level of curves, $\overline{M}_{0,n}$ behaves like a Fano variety, even though $\overline{M}_{0,n}$ is Fano only for $n \leq 5$. Moreover, by the Bridge Theorem of [25], the F-conjecture for $\overline{M}_{0,g+n}$ implies the analogous result for $\overline{M}_{g,n}$, yet for $g = 22$ ([19]) and $g \geq 24$ ([30] and [17]), $\overline{M}_g$ is of general type, i.e. in some sense as far as possible from being Fano. Recent work of Gibney, however, has enabled a computer-assisted proof of the F-conjecture for $\overline{M}_g$ for $g \leq 24$ ([24]).

We now describe our proof, which uses techniques from both algebraic and convex geometry. A main obstacle to proving Fulton's conjecture is the lack of a canonical basis for $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$. In Section 2.2, we define an $\binom{n}{4}$-dimensional subspace of $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$, called the *Keel subspace*, that admits a canonical basis, meaning that every F-divisor class written in this basis has a natural representative as an effective sum of boundary divisors. This result holds for all n; see Theorem 2.2.5. In the notation of Conjecture 2.1.2, there is a section σ of $\phi : V \to N^1(\overline{M}_{0,n})_{\mathbb{Q}}$ such that $\sigma(\mathcal{F})$ is contained in the positive orthant $\langle \Delta_{J,J^c} \rangle_{\geq 0}$. In particular, for F-nef classes in the Keel subspace, we exhibit both an obvious choice of representative divisor and a recipe for combining F-inequalities to prove non-negativity of all boundary coefficients. Our approach for Fulton's conjecture is extend to a basis of $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$ so that this choice of divisor and recipe for combining inequalities give every F-nef divisor in $\overline{M}_{0,n}$ as an effective sum of boundary. For both $n = 6$ and 7, this approach proves the conjecture for a codimension one subspace of $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$; see Corollary 2.2.6 and Proposition 2.4.3. For an F-nef divisor D outside of this subspace, some of the coefficients of boundary divisors can be negative, so to finish the proof of Fulton's conjecture we exploit the symmetry of Keel relations to find a different representative of D as an effective sum of boundary divisors. We determine which inequalities establish non-negativity of this new representative by hand for $n = 6$ (see Lemma 2.3.3 and Corollary 2.3.4) and via the simplex algorithm for $n = 7$. We give these inequalities as intersections with explicit sums of F-curves. Even for $n = 7$ it is then straightforward to verify by hand that the new representative is an effective sum of boundary divisors. We hope that this approach will extend to $n > 7$, but for for higher n it is likely impracticable to give inequalities as explicit sums of F-curves as is done in the Appendix (Section 2.5).

To conclude this section, we make a few remarks on notation. The first matter deserving comment is our choice of name for the conjecture. We follow [20] and [21] in using 'Fulton's conjecture' to mean a modified version of Fulton's original question from [39]. We use without further mention that rational and numerical equivalence coincide for $\overline{M}_{0,n}$ ([38]); for notational simplicity we refer mostly to numerical equivalence. By the 'obvious representative' for a divisor class δ satisfying $\delta = [D]$, we mean the choice of representative D (recall from Definition 1.1.1 that brackets around a cycle indicate its numerical equivalence class).

To define certain sums of boundary divisors, we temporarily fix the following labeling scheme for a boundary divisor Δ_J: we determine Δ_J uniquely by taking $J \subseteq \{1, \ldots, n\}$ such that $1 \leq |J| \leq n/2$, and, in case $|J| = n/2$, with $1 \in J$. For each j such that $2 \leq j \leq n/2$, we define the sum

$$B_j = \sum_{|J|=j} \Delta_J. \tag{2.1.1}$$

At the end of Section 2.2, we use a different labeling convention to facilitate calculations involving Kapranov's blow-up construction of $\overline{M}_{0,n}$.

2.2 Fulton's conjecture for Keel classes

The main actors in this chapter are divisors obtained by averaging the obvious representatives of the Keel relations among boundary divisors described in Equation 1.3.1 of Chapter 1. For each $n \geq 4$, we define the *Keel divisor* S_I as follows: let $I = \{i, j, k, l\} \subseteq \{1, \ldots, n\}$ and, recalling Notation 1.2.6, set

$$S_I = \frac{1}{3}\Bigg(\sum_{\substack{\{i,j\} \subseteq T, \\ \{k,l\} \subseteq T^c}} \Delta_T + \sum_{\substack{\{i,k\} \subseteq T, \\ \{j,l\} \subseteq T^c}} \Delta_T + \sum_{\substack{\{i,l\} \subseteq T, \\ \{j,k\} \subseteq T^c}} \Delta_T \Bigg). \tag{2.2.1}$$

The class of a Keel divisor is denoted $[S_I]$, and is called a *Keel class*. Note that these are defined as $\mathbb{Q}$-divisors, but on the level of numerical equivalence, we could take any of the three summands as our definition. The particular choice of S_I in Equation (2.2.1) is important for finding a representative in a given numerical equivalence class, all of whose boundary divisor coefficients are non-negative.

Lemma 2.2.1. *The $\binom{n}{4}$ Keel classes $[S_I]$ are linearly independent in $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$.*

The intuition behind this lemma is that any non-trivial linear relation among the Keel classes would give a relation in $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$ in addition to the Keel relations, contradicting that the Keel relations generate all relations among boundary divisors. Rather than making this idea precise, we defer the proof to the discussion at the end of the present section, where it appears as Corollary 2.2.8.

Keel classes exhibit very nice intersection properties with respect to F-curves. These intersections can be determined by standard calculations in $\overline{M}_{0,n}$ (see Section 1.3), or more directly via intersection theory on $\mathbb{P}^1$.

Lemma 2.2.2. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ be a partition of $\{1, \ldots, n\}$, and let C be the corresponding F-curve (see Notation 1.3.3). If $a_I = |\mathcal{A} \cap I|$, and similarly for b_I, c_I and d_I,* *then*

$$[S_I] \cdot C = \begin{cases} 1 & \text{if } a_I = b_I = c_I = d_I = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\pi_I : \overline{M}_{0,n} \to \overline{M}_{0,4} \cong \mathbb{P}^1$ be the forgetful morphism that remembers only the marked points labeled by I. It is easily checked that, if $I = \{i, j, k, l\}$,

$$S_I = \pi_I^*\left(\frac{1}{3}(\Delta_{ij} + \Delta_{ik} + \Delta_{il})\right). \tag{2.2.2}$$

Since all points of $\mathbb{P}^1$ are numerically equivalent, it follows that $[S_I] = \pi_I^*([pt])$. By the projection formula, $[S_I] \cdot [C] = \pi_I^*([pt]) \cdot [C] = ([pt]) \cdot ((\pi_I)_*([C]))$. Finally, $(\pi_I)_*([C]) = [\overline{M}_{0,4}] = [\mathbb{P}^1]$ precisely when $a_I = b_I = c_I = d_I = 1$, and is 0 otherwise. □

Let $n \geq 4$ and let $\mathcal{K}_n$ be the $\binom{n}{4}$-dimensional subspace of $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$ generated by the Keel classes. We call $\mathcal{K}_n$ the *Keel subspace* of $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$.

Lemma 2.2.3. *The coefficient c_J of Δ_J in $D = \sum s_I S_I = \sum c_J \Delta_J$ is*

$$c_J = \frac{1}{3} \sum_{|I \cap J| = 2} s_I.$$

Proof. By definition, each Δ_J with $|I \cap J| = 2$ appears with multiplicity $1/3$ in S_I. □

Example 2.2.4. To illustrate notation, we consider the well-known example of $\overline{M}_{0,5}$. The Keel divisors here are

$$\begin{aligned}
S_{1234} &= \frac{1}{3}((\Delta_{12} + \Delta_{34}) + (\Delta_{13} + \Delta_{24}) + (\Delta_{14} + \Delta_{23})), \\
S_{1235} &= \frac{1}{3}((\Delta_{12} + \Delta_{35}) + (\Delta_{13} + \Delta_{25}) + (\Delta_{15} + \Delta_{23})), \\
S_{1245} &= \frac{1}{3}((\Delta_{12} + \Delta_{45}) + (\Delta_{14} + \Delta_{25}) + (\Delta_{15} + \Delta_{24})), \\
S_{1345} &= \frac{1}{3}((\Delta_{13} + \Delta_{45}) + (\Delta_{14} + \Delta_{35}) + (\Delta_{15} + \Delta_{34})), \\
S_{2345} &= \frac{1}{3}((\Delta_{23} + \Delta_{45}) + (\Delta_{24} + \Delta_{35}) + (\Delta_{25} + \Delta_{34})).
\end{aligned}$$

If $D = \sum s_I S_I$, the coefficient of Δ_{12} in D is

$$c_{12} = \frac{1}{3}(s_{1234} + s_{1235} + s_{1245}) = \frac{1}{3} \sum_{|I \cap \{1,2\}| = 2} s_I.$$

Theorem 2.2.5 (Fulton's conjecture for $\mathcal{K}_n$)**.** *If $[D] \in \mathcal{K}_n$ is an F-nef divisor class, then $D = \sum s_I S_I$ is an effective sum of boundary divisors.*

Proof. We show that, for each boundary divisor index J, and each four-tuple $\mathbf{t} = (n_1, n_2, n_3, n_4) \in \mathbb{N}^4$ with $n_1 + n_2 + n_3 + n_4 = n$, there exists a collection of F-curves,

$F_J^{\mathbf{t}}$, and a positive integer $m_{\mathbf{t}}$, such that

$$D \cdot \sum_{C \in F_J^{\mathbf{t}}} C = m_{\mathbf{t}} c_J.$$

Recall from Definition 1.3.4 that a four-tuple $\mathbf{t} = (n_1 : n_2 : n_3 : n_4)$ is called the *type* of an F-curve, where each n_i gives the cardinality of the i^{th} element of a partition determining the F-curve. Up to a possible reordering of partitions, define

$$F_J^{\mathbf{t}} = \{C(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) : C \text{ of type } \mathbf{t},\ \mathcal{A} \cup \mathcal{B} = J\}.$$

We claim that $F_J^{\mathbf{t}}$ is the set of F-curves C of type $\mathbf{t}$ such that $C \cdot S_I = 0$ unless $|I \cap J| = 2$. In other words, $F_J^{\mathbf{t}}$ consists of all F-curves of type $\mathbf{t}$ whose intersection with D gives inequalities involving only the s_I that appear in c_J as in Lemma 2.2.3.

To prove this claim, suppose first that C is of type $\mathbf{t}$, and that there exists an S_I such that $C \cdot S_I = 1$ with $|I \cap J| \neq 2$. Using the notation of Lemma 2.2.2, $a_I = b_I = c_I = d_I = 1$, so $|I \cap J| \neq 2$ implies that either there are elements of J^c in $\mathcal{A}$ or $\mathcal{B}$ (for $|I \cap J| < 2$), or there are elements of J in $\mathcal{C}$ or $\mathcal{D}$ (for $|I \cap J| > 2$), hence $C \notin F_J^{\mathbf{t}}$.

Conversely, if C is of type $\mathbf{t}$ and there are elements of J in more than two of the partitions, then we can find a four-tuple I with $|I \cap J| > 2$ and $C \cdot S_I = 1$. Hence we may assume that J is contained in two of the partitions, $\mathcal{A}$ and $\mathcal{B}$, for example. If there is an element of J^c in $\mathcal{A} \cup \mathcal{B}$, then there is an I with $|I \cap J| \leq 1$, but $C \cdot S_I = 1$, hence proving the claim.

To calculate $m_{\mathbf{t}}$, select a term s_I in the sum c_J. We count how many F-curves $C \in F_J^{\mathbf{t}}$ intersect S_I with value 1. Set $\{i, j\} = I \cap J$ (since s_I appears in c_J, we know $|I \cap J| = 2$), and $\mathcal{A} \cup \mathcal{B} = J$. If $|\mathcal{A}| = |\mathcal{B}|$, there is no loss of generality in assuming $i \in \mathcal{A}$ and $j \in \mathcal{B}$. Then the remaining elements of the partitions $\mathcal{A}$ and $\mathcal{B}$ can be chosen in $\binom{|J|-2}{|\mathcal{A}|-1}$ ways. If $|\mathcal{A}| \neq |\mathcal{B}|$, we must also consider $i \in \mathcal{B}$ and $j \in \mathcal{A}$, giving in total $2\binom{|J|-2}{|\mathcal{A}|-1}$ ways to choose $\mathcal{A}$ and $\mathcal{B}$. Similarly, there are $\binom{n-|J|-2}{|\mathcal{C}|-1}$ ways to choose $\mathcal{C}$ and $\mathcal{D}$ if $|\mathcal{C}| = |\mathcal{D}|$, and $2\binom{n-|J|-2}{|\mathcal{C}|-1}$ ways if $|\mathcal{C}| \neq |\mathcal{D}|$.

Hence for each s_I in c_J, precisely $m' = (2 - \delta_{|\mathcal{A}|,|\mathcal{B}|})(2 - \delta_{|\mathcal{C}|,|\mathcal{D}|})\binom{|J|-2}{|\mathcal{A}|-1}\binom{n-|J|-2}{|\mathcal{C}|-1}$ curves $C \in F_J^{\mathbf{t}}$ intersect S_I with value 1, where $\delta_{x,y}$ is the Kronecker-delta symbol. Therefore

$$D \cdot \sum_{C \in F_J^{\mathbf{t}}} C = m' \sum_{|I \cap J| = 2} s_I = 3\, m' c_J,$$

so we see $m_{\mathbf{t}} = 3\, m'$. Since D is an F-nef divisor, and $m_{\mathbf{t}} \geq 0$, it follows that $c_J \geq 0$. $\square$

Example 2.2.4, continued. There is only one type of F-curve in $\overline{M}_{0,5}$, so to prove that $c_{12} \geq 0$, we consider

$$F_{12}^{(1:1:1:2)} = \left\{C\left(1, 2, 3, \tfrac{4}{5}\right), C\left(1, 2, 4, \tfrac{3}{5}\right), C\left(1, 2, 5, \tfrac{3}{4}\right)\right\}.$$

For $D = \sum s_I S_I$, the corresponding F-inequalities are

$$\begin{aligned} D \cdot C(1,2,3,\tbinom{4}{5}) &= s_{1234} + s_{1235} \geq 0, \\ D \cdot C(1,2,4,\tbinom{3}{5}) &= s_{1234} + s_{1245} \geq 0, \\ D \cdot C(1,2,5,\tbinom{3}{4}) &= s_{1235} + s_{1245} \geq 0. \end{aligned}$$

Hence if $[D]$ is an F-nef divisor,

$$D \cdot \sum_{C \in F_{12}^{(1:1:1:2)}} C = 2\,(s_{1234} + s_{1235} + s_{1245}) = 6\,c_{12} \geq 0.$$

Corollary 2.2.6. *Fulton's conjecture is true for $\overline{M}_{0,5}$ and for a codimension one subspace of $N^1(\overline{M}_{0,6})$.*

Proof. The Keel subspace has dimension $\binom{n}{4}$, while $\dim N^1(\overline{M}_{0,n}) = 2^{n-1} - \binom{n}{2} - 1$. $\square$

In particular, Theorem 2.2.5 gives less information as n increases. In Sections 2.3 and 2.4, however, we complete the proof of Fulton's conjecture for $n = 6$ and 7.

To conclude this section, we relate the Keel subspace to Kapranov's construction of $\overline{M}_{0,n}$ as an iterated blow-up of $\mathbb{P}^{n-3}$ along linear centers (see Section 1.3.3). For $n \geq 6$, the last two stages of blow-ups in the Kapranov construction produce $\binom{n-1}{n-5} + \binom{n-1}{n-4} = \binom{n}{4}$ exceptional divisors, so it is natural to ask how the Keel subspace relates to the subspace of $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$ spanned by exceptional divisor classes from these last two stages of blow-ups.

Let $\mathrm{E}(j)$ be the set of exceptional divisor classes from the j^{th} stage of the Kapranov construction, let $[H]$ be the pull-back of the hyperplane class from $\mathbb{P}^{n-3}$, and let $\mathrm{Exc}(k)$ be the subspace of $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$ generated by $[H]$ and $\mathrm{E}(j)$ for $j \leq k$.

Proposition 2.2.7. *Let $\pi_{\mathrm{Exc}(n-6)} : N^1(\overline{M}_{0,n})_{\mathbb{Q}} \to N^1(\overline{M}_{0,n})_{\mathbb{Q}}/\mathrm{Exc}(n-6)$ be the projection map. Then $\pi_{\mathrm{Exc}(n-6)}$ restricted to the Keel subspace is an isomorphism.*

Corollary 2.2.8. *The Keel classes are linearly independent in $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$.*

Proof. $N^1(\overline{M}_{0,n})_{\mathbb{Q}}/\mathrm{Exc}(n-6)$ is isomorphic to the subspace of $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$ generated by $\mathrm{E}(n-5)$ and $\mathrm{E}(n-4)$. Thus $\dim(\mathcal{K}_n) = \binom{n}{4}$, which equals the number of Keel classes. $\square$

Corollary 2.2.9. *The class $[H]$ plus the classes of divisors from the first $n-6$ stages of the Kapranov construction extend the Keel classes to a basis of $N^1(\overline{M}_{0,n})_{\mathbb{Q}}$ for $n \geq 6$.*

The proof of Proposition 2.2.7 will follow by showing a particular intersection matrix is full rank, where all intersections are described in the next lemma. For the remainder of this section, we use a different indexing scheme for boundary divisors occurring as exceptional divisors in the Kapranov construction. Namely, we write each such divisor uniquely as Δ_J with $3 \leq |J| \leq n-2$ and $n \notin J$. For example, elements of $\mathrm{E}(n-5)$ are written as $[\Delta_J]$ where $n \notin J$ and $|J| = 4$, and likewise elements of $\mathrm{E}(n-4)$ are $[\Delta_J]$, where $n \notin J$ and $|J| = 3$. We denote elements of the dual Kapranov basis by $[\Delta_J]^\vee$ and

$H^\vee$. For present purposes, we need only the intersections $[\Delta_J]^\vee \cdot [S_I]$ for $3 \le |J| \le 4$, but with no additional effort, we can calculate the intersections of Keel divisors with all 1-cycles $[\Delta_J]^\vee$.

Lemma 2.2.10. *Let $n \notin J$ with $3 \le |J| \le n-2$. Then $[\Delta_J]^\vee \cdot [S_I] = \min\{0, 2-|I \cap J|\}$.*

Proof. We first consider how one-cycles of the dual Kapranov basis for $N_1(\overline{M}_{0,n})_{\mathbb{Q}}$ intersect the Δ_{ij} with $n \notin \{i,j\}$. Note that these are the only boundary divisors not appearing as exceptional divisors in the Kapranov construction. We write such boundary divisors in the Kapranov basis as

$$[\Delta_{ij}] = [H] - \sum_{\substack{J \supseteq \{i,j\}, n \notin J, \\ 3 \le |J| \le n-2}} [\Delta_J]. \tag{2.2.3}$$

It follows that the intersections $[\Delta_J]^\vee \cdot \Delta_{ij}$ are given by -1 times the number of times $[\Delta_J]$ appears in the right hand side of Equation (2.2.3), that is,

$$[\Delta_J]^\vee \cdot \Delta_{ij} = \begin{cases} 0 & \text{if } \{i,j\} \not\subseteq J, \\ -1 & \text{if } \{i,j\} \subseteq J. \end{cases}$$

Now it is easy to calculate the intersection $[\Delta_J]^\vee \cdot S_I$: it is $1/3$ times the number of times Δ_J appears in S_I minus the number of times Δ_{ij} appears in S_I, where $\{i,j\} \subseteq J$, i.e.

$$\begin{aligned} [\Delta_J]^\vee \cdot S_I &= \frac{1}{3}\left(\left\{ \begin{array}{ll} 1 & \text{if } |I \cap J| = 2 \\ 0 & \text{if } |I \cap J| \neq 2 \end{array} \right\} - \big|\{\{i,j\} : \{i,j\} \subseteq I \cap J\}\big| \right) \\ &= \frac{1}{3}\left(\left\{ \begin{array}{ll} 1 & \text{if } |I \cap J| = 2 \\ 0 & \text{if } |I \cap J| \neq 2 \end{array} \right\} - \binom{|I \cap J|}{2} \right). \end{aligned} \tag{2.2.4}$$

Since $0 \le |I \cap J| \le 4$, by enumerating the five possibilities, we obtain precisely the desired formula. □

Proof of Proposition 2.2.7. We can write each Keel class as a linear combination of classes in the Kapranov basis by calculating the intersection matrix given by pairing the Keel classes and the dual Kapranov basis. The projection $\pi_{\mathrm{Exc}(n-6)}$ restricted to the Keel subspace is given by setting intersections with all but the 1-cycles dual to $\mathrm{E}(n-5)$ and $\mathrm{E}(n-4)$ to zero, i.e. by the the matrix pairing Keel classes and the 1-cycles dual to $\mathrm{E}(n-5)$ and $\mathrm{E}(n-4)$. The proposition will follow by showing that the determinant of this matrix is non-zero.

Let M be the intersection matrix of Keel classes with 1-cycles dual to $\mathrm{E}(n-5)$ and $\mathrm{E}(n-4)$. M can be decomposed into the following blocks:

$$M = \begin{pmatrix} A = ([\Delta_J]^\vee \cdot [S_I])_{\substack{n \notin I \cup J, \\ |J|=4}} & B = ([\Delta_J]^\vee \cdot [S_I])_{\substack{n \in I - J, \\ |J|=4}} \\ C = ([\Delta_J]^\vee \cdot [S_I])_{\substack{n \notin I \cup J, \\ |J|=3}} & D = ([\Delta_J]^\vee \cdot [S_I])_{\substack{n \in I - J, \\ |J|=3}} \end{pmatrix},$$

so the rows of this matrix are indexed by k-tuples J, $3 \leq k \leq 4$, while the columns are indexed by four-tuples I. We claim that $\det(M) = 2^{\binom{n-1}{4}}(-1)^{\binom{n-1}{3}}$. In particular, M is nonsingular.

By Lemma 2.2.10, $D = \mathrm{diag}(-1, \ldots, -1)$, and hence

$$\det(M) = \det(A - BD^{-1}C)\det(D) = (-1)^{\binom{n-1}{3}}\det(A + BC).$$

The rows of BC are labeled by indices J with $n \notin J$, $|J| = 4$, while the columns are indexed by sets I with $n \notin I$. With this labeling scheme,

$$(BC)_{J,I} = \sum_{\substack{n \notin J', \\ |J'|=3}} ([\Delta_J]^\vee \cdot [S_{J' \cup \{n\}}])([\Delta_{J'}]^\vee \cdot [S_I]).$$

Since J' of the sum has cardinality three, and since $n \notin J$, $|I \cap J'|$ and $|(J' \cup \{n\}) \cap J|$ are at most three, by Lemma 2.2.10 both $[\Delta_J]^\vee \cdot [S_{J' \cup \{n\}}]$ and $[\Delta_{J'}]^\vee \cdot [S_I]$ are greater than -2, and their product is non-zero precisely when both equal -1. Therefore

$$\begin{aligned}(BC)_{J,I} &= |\{J' : |J' \cap (I \cap J)| = 3\}| \\ &= \begin{cases} 4 & \text{if } I = J, \\ 1 & \text{if } |I \cap J| = 3, \\ 0 & \text{if } |I \cap J| \leq 2, \end{cases}\end{aligned}$$

while

$$\begin{aligned}(A)_{J,I} &= \min\{0, 2 - |I \cap J|\} \\ &= \begin{cases} -2 & \text{if } I = J, \\ -1 & \text{if } |I \cap J| = 3, \\ 0 & \text{if } |I \cap J| \leq 2. \end{cases}\end{aligned}$$

Hence $A + BC = diag(2, \ldots, 2)$, proving the claim and the proposition. □

2.3 Fulton's conjecture for $\overline{M}_{0,6}$

Theorem 2.2.5 provides a proof of Fulton's conjecture for a codimension one subspace of $N^1(\overline{M}_{0,6})_{\mathbb{Q}}$. The initial strategy for $\overline{M}_{0,6}$ is to extend the proof for the Keel subspace to all of $N^1(\overline{M}_{0,6})_{\mathbb{Q}}$. Concretely, we pick a divisor B depending on some parameters such that the Keel classes and the class of B give a basis for $N^1(\overline{M}_{0,6})_{\mathbb{Q}}$. This gives one condition on our parameters. For an arbitrary divisor class $[D]$, we pick the representative

$$D = \sum s_I S_I + bB.$$

Next we consider for each type of F-curve $\mathbf{t} = (n_1 : n_2 : n_3 : n_4)$ the inequalities arising by intersecting D with the F-curves of the collection $F_J^{\mathbf{t}}$ of the proof of Theorem 2.2.5. The resulting inequalities for c_J, the coefficient of Δ_J in D, depend now on the parameters that define B. The idea is to pick these parameters so that $c_J \geq 0$ while avoiding the parameters that would put $[B]$ in the Keel subspace.

We extend the Keel classes to a basis of $N^1(\overline{M}_{0,6})_{\mathbb{Q}}$ by adjoining the divisor class of $B_{\lambda,\mu} = \lambda B_2 + \mu B_3$, where B_2 and B_3 are sums of boundary divisors defined in Equation 2.1.1.

Lemma 2.3.1. *The Keel classes and $[B_{\lambda,\mu}]$ form a basis of $N^1(\overline{M}_{0,6})_{\mathbb{Q}}$ if and only if $3\lambda - 2\mu \neq 0$.*

Proof. Corollary 2.2.9 states that the Keel classes plus the hyperplane class form a basis for $N^1(\overline{M}_{0,6})_{\mathbb{Q}}$, so to ensure $[B_{\lambda,\mu}]$ completes the Keel classes to a basis, we calculate which λ and μ guarantee that the projection map $\pi_{\mathcal{K}_6} : N^1(\overline{M}_{0,6})_{\mathbb{Q}} \to N^1(\overline{M}_{0,6})_{\mathbb{Q}}/\mathcal{K}_6$ restricted to the span of $[B_{\lambda,\mu}]$ give an isomorphism. Concretely, we write $[B_{\lambda,\mu}]$ in terms of the basis of Corollary 2.2.9, and check which values of λ and μ correspond to a non-zero coordinate for the hyperplane class.

The hyperplane class in the Kapranov model is the psi-class of the 'moving point' ([36]), so, for the usual choice of the last marked point as the moving point, $[H] = \psi_6$. Suppose we can write $[B_2]$ and $[B_3]$ in the following form:

$$[B_2] = \alpha_2 \sum_{6 \notin I} [S_I] + \beta_2 \sum_{6 \in I} [S_I] + h_2[H], \tag{2.3.1}$$

$$[B_3] = \alpha_3 \sum_{6 \notin I} [S_I] + \beta_3 \sum_{6 \in I} [S_I] + h_3[H]. \tag{2.3.2}$$

We simultaneously determine α_i, β_i, and h_i and prove the above suppositions true by intersecting both left and right sides of the proposed equalities with all F-curves of $\overline{M}_{0,6}$ to obtain a consistent (and over-determined) system of linear equations in α_i, β_i, and h_i. It would suffice here to consider intersections with any 16 1-cycles forming a basis of $N_1(\overline{M}_{0,6})_{\mathbb{Q}}$, but due to symmetry, the approach employed below results in fewer equations to be solved.

The intersections of $[B_2]$ and $[B_3]$ with F-curves can be either calculated as in Section 1.3: for C a (1:1:1:3) F-curve and C' a (1:1:2:2) F-curve,

$$\begin{aligned} &[B_2] \cdot C = 3, \quad [B_2] \cdot C' = -1, \\ &[B_3] \cdot C = -1, \ [B_3] \cdot C' = 2. \end{aligned} \tag{2.3.3}$$

The intersections of the right hand side of proposed equations (2.3.1) and (2.3.2) with F-curves C and C' follow from Lemma 2.2.2 and the fact that $H = \psi_6$ intersects an F-curve with value 1 if the marked point 6 is on the spine, and with value 0 if the marked point 6 is on a tail. The resulting system of equations for Equation (2.3.1) is

$$(*) \begin{cases} 3 &= 2\alpha_2 + \beta_2 \quad \text{for } C \text{ a } (1:1:1:3) \text{ curve with 6 on its tail,} \\ 3 &= 3\beta_2 + h_2 \quad \text{for } C \text{ a } (1:1:1:3) \text{ curve with 6 on its spine,} \\ -1 &= 2\alpha_2 + 2\beta_2 \quad \text{for } C' \text{ a } (1:1:2:2) \text{ curve with 6 on its tail,} \\ -1 &= 4\beta_2 + h_2 \quad \text{for } C' \text{ a } (1:1:2:2) \text{ curve with 6 on its spine.} \end{cases}$$

The analogous system for Equation (2.3.2) results from replacing the left hand side of (∗) with the appropriate intersections with $[B_3]$, and the parameters on the right hand side with α_3, β_3, and h_3.

The system of linear equations for α_i, β_i, and h_i, $i = 2, 3$, can be simultaneously solved by Gauss-Jordan elimination, giving

$$\begin{aligned}[B_2] &= \frac{7}{2}\sum_{6\notin I}[S_I] - 4\sum_{6\in I}[S_I] + 15[H], \\ [B_3] &= -2\sum_{6\notin I}[S_I] + 3\sum_{6\in I}[S_I] - 10[H].\end{aligned}$$

The image of $\lambda[B_2] + \mu[B_3]$ under the projection map $\pi_{\mathcal{K}_6}$ is $(15\lambda - 10\mu)H$, hence $[B_{\lambda,\mu}]$ completes the Keel classes to a basis precisely when $3\lambda - 2\mu \neq 0$. ☐

Suppose that $3\lambda - 2\mu \neq 0$. Every divisor class $[D]$ can be written uniquely as

$$[D] = \sum s_I[S_I] + b[B_{\lambda,\mu}]. \tag{2.3.4}$$

By Lemma 2.2.2 and Equations (2.3.3), the F-inequalities from intersecting $[D]$ with an F-curve $C(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ in the respective cases $|\mathcal{A}| = |\mathcal{B}| = |\mathcal{C}| = 1, |\mathcal{D}| = 3$ and $|\mathcal{A}| = |\mathcal{B}| = 1, |\mathcal{C}| = |\mathcal{D}| = 2$ are of the form

$$\begin{aligned}&s_{I_1} + s_{I_2} + s_{I_3} + (3\lambda - \mu)\, b \geq 0, \\ &s_{I'_1} + s_{I'_2} + s_{I'_3} + s_{I'_4} + (-\lambda + 2\mu)\, b \geq 0,\end{aligned}$$

where, in the notation of Lemma 2.2.2, the index sets I_i (respectively I'_j) in each inequality represent all possibilities for $a_{I_i} = b_{I_i} = c_{I_i} = d_{I_i} = 1$ (respectively $a_{I'_j} = \ldots = d_{I'_j} = 1$).

Examples of each type of inequality are

$$\begin{aligned}&s_{1234} + s_{1235} + s_{1236} + (3\lambda - \mu)\, b \geq 0, \\ &s_{1235} + s_{1236} + s_{1245} + s_{1246} + (-\lambda + 2\mu)\, b \geq 0.\end{aligned}$$

Theorem 2.3.2 (Fulton's conjecture for $\overline{M}_{0,6}$)**.** *Let $[D]$ be an F-divisor class in $\overline{M}_{0,6}$. Then D is linearly equivalent to an effective sum of boundary divisors.*

Proof. We assume that $[D]$ is an F-divisor, and choose the obvious representative from Equation (2.3.4), namely,

$$D = \sum s_I S_I + b\, B_{\lambda,\mu}. \tag{2.3.5}$$

By symmetry, it suffices to prove that c_{12} and c_{123}, the coefficients of Δ_{12} and Δ_{123} in

Equation (2.3.5), respectively, are nonnegative. By Lemma 2.2.3,

$$c_{12} = \frac{1}{3}(s_{1234} + s_{1235} + s_{1236} + s_{1245} + s_{1246} + s_{1256}) + \lambda\, b, \tag{2.3.6}$$

$$\begin{aligned} c_{123} = \frac{1}{3}(&s_{1245} + s_{1246} + s_{1256} + s_{1345} + s_{1346} \\ &+ s_{1356} + s_{2345} + s_{2346} + s_{2356}) + \mu\, b. \end{aligned} \tag{2.3.7}$$

To show $c_{12} \geq 0$, we proceed as in the proof of Theorem 2.2.5: we fix a type of F-curve, $(1:1:1:3)$, for example, and consider the collection of F-curves

$$F_{12}^{(1:1:1:3)} = \{C(1,2,3,\substack{4\\5\\6}), C(1,2,4,\substack{3\\5\\6}), C(1,2,5,\substack{3\\4\\6}), C(1,2,6,\substack{3\\4\\5})\}.$$

By the proof of Theorem 2.2.5, we obtain

$$D \cdot \sum_{C \in F_{12}^{(1:1:1:3)}} C = 2 \sum_{\{1,2\} \subseteq I} s_I + 4(3\lambda - \mu)b.$$

Substituting $6c_{12} = 2\sum_{\{1,2\}\subseteq I} s_I + 6\lambda b$, and noting that $[D]$ is an F-divisor,

$$6c_{12} + 2(3\lambda - 2\mu)b \geq 0. \tag{2.3.8}$$

We may not set $3\lambda - 2\mu = 0$, but there is another inequality for c_{12} arising from the collection $F_{12}^{(1:1:2:2)} = \{C(1,2,\substack{3\\4},\substack{5\\6}), C(1,2,\substack{3\\5},\substack{4\\6}), C(1,2,\substack{3\\6},\substack{4\\5})\}$, giving

$$D \cdot \sum_{C \in F_{12}^{(1:1:2:2)}} C = 2 \sum_{\{1,2\} \subseteq I} s_I + 3(-\lambda + 2\mu)b = 6c_{12} + 3(-3\lambda + 2\mu)b \geq 0.$$

Combining the two inequalities yields

$$D \cdot \left(\frac{1}{10} \sum_{C \in F_{12}^{(1:1:1:3)}} C + \frac{1}{15} \sum_{C' \in F_{12}^{(1:1:2:2)}} C' \right) = c_{12} \geq 0.$$

We cannot conclude in the same way that $c_{123} \geq 0$, but the resulting inequality is nevertheless useful. Only the $(1:1:2:2)$ F-curves can have two of their partitions equal to $\{1,2,3\}$. We order the partitions so that $\mathcal{B} \cup \mathcal{C} = \{1,2,3\}$ to obtain

$$\begin{aligned} F_{123}^{(1:1:2:2)} = \{&C(4,1,\substack{2\\3},\substack{5\\6}), C(4,2,\substack{1\\3},\substack{5\\6}), C(4,3,\substack{1\\2},\substack{5\\6}), C(5,1,\substack{2\\3},\substack{4\\6}), C(5,2,\substack{1\\3},\substack{4\\6}), \\ &C(5,3,\substack{1\\2},\substack{4\\6}), C(6,1,\substack{2\\3},\substack{4\\5}), C(6,2,\substack{1\\3},\substack{4\\5}), C(6,3,\substack{1\\2},\substack{4\\5})\}. \end{aligned}$$

As above,

$$D \cdot \sum_{C \in F_{123}^{(1:1:2:2)}} C = 4 \sum_{|I \cap \{1,2,3\}| = 2} s_I + 9(-\lambda + 2\mu)b.$$

Substituting $12c_{123} = 4\sum_{|I\cap\{1,2,3\}|=2} s_I + 12\mu b$ results in the inequality

$$12c_{123} + 3(-3\lambda + 2\mu)b \geq 0, \tag{2.3.9}$$

or equivalently $4c_{123} \geq (3\lambda - 2\mu)b$. Since the proof of Theorem 2.2.5 gives no additional inequalities for c_{123}, and since we may not choose λ, μ such that $3\lambda - 2\mu = 0$, instead we pick $(\lambda, \mu) = (\frac{2}{5}, \frac{1}{5})$, that is, we complete to a basis with the anticanonical class $-K_{\overline{M}_{0,6}}$. For this choice, $3\lambda - 2\mu = 4/5$, so we need only consider $b < 0$.

Lemma 2.3.3. *For D an F-divisor in $\overline{M}_{0,6}$ written as in Equation (2.3.5), at most one of the c_{1kl} is negative.*

Proof. Without loss of generality, we may suppose for a contradiction that c_{123} and c_{124} are negative. In terms of the boundary coefficients of D as in Equation (2.3.5), the intersection of D with the F-curve $C(3, 4, \frac{1}{2}, \frac{5}{6})$ gives

$$c_{123} + c_{124} - c_{12} - c_{56} + c_{34} \geq 0. \tag{2.3.10}$$

We will obtain a contradiction by proving that $c_{12} + c_{56} - c_{34} \geq 0$.

By equation (2.3.6),

$$\begin{aligned} c_{12} + c_{56} - c_{34} = \frac{1}{3}(&s_{1235} + s_{1236} + s_{1245} + s_{1246} + s_{1356} + s_{1456} \\ &+ s_{2356} + s_{2456} - s_{1345} - s_{1346} - s_{2345} - s_{2346}) \\ &+ \frac{2}{3}s_{1256} + \frac{2}{5}b, \end{aligned}$$

while the negativity of c_{123} and c_{124} implies that

$$\begin{aligned} &-\frac{1}{3}(s_{1345} + s_{1346} + s_{2345} + s_{2346}) \\ &\quad > \frac{1}{6}(s_{1245} + s_{1246} + s_{2356} + s_{1235} + s_{1236} + s_{2456}) + \frac{1}{3}\, s_{1256} + \frac{1}{5}\, b, \end{aligned}$$

and so

$$\begin{aligned} &c_{12} + c_{56} - c_{34} \\ &\quad > \frac{3}{5}b + s_{1256} + \frac{1}{2}(s_{1235} + s_{1236} + s_{1245} + s_{1246} + s_{1356} + s_{1456} + s_{2356} + s_{2456}). \end{aligned}$$

The right hand side of this inequality can be rewritten in terms of the intersections of D with the F-curves $C(1, 2, 5, \substack{3\\4\\6})$, $C(1, 2, 6, \substack{3\\4\\5})$, and $C(5, 6, \frac{1}{2}, \frac{3}{4})$ to give the desired contradiction: $c_{12} + c_{56} - c_{34} > -\frac{2}{5}b > 0$. □

Corollary 2.3.4. *If D is an F-divisor, then the coefficients c_{1jk} and c_{1mn} with $(j, k) \neq (m, n)$ satisfy $c_{1jk} + c_{1mn} \geq 0$.*

Next we 'spread out the negativity' of c_{123} with the assistance of an average expression for $-[\Delta_{123}]$ from [21]:

$$-[\Delta_{123}] = -\frac{1}{9}\sum_{\substack{i,j\in\{2,\ldots,6\},\\(i,j)\neq(2,3)}}[\Delta_{1ij}] - \frac{2}{9}\sum_{\substack{a\in\{1,2,3\},\\x\in\{4,5,6\}}}[\Delta_{ax}]$$
$$+\frac{1}{3}\sum_{a,b\in\{1,2,3\}}[\Delta_{ab}] + \frac{1}{3}\sum_{x,y\in\{4,5,6\}}[\Delta_{xy}].$$

We substitute the obvious representative of $-\Delta_{123}$ from this average into D. Denote the coefficients in this new expression by c'_{ij} and c'_{1kl} (in particular, $c'_{123}=0$).

To conclude the proof, it remains to show that all c'_{ij} and c'_{1kl} are nonnegative. By inequalities (2.3.8) and (2.3.9),

$$c_{ij} \geq -\frac{4}{15}b \text{ and } c_{123} \geq \frac{1}{5}b,$$

so by the above average for $-\Delta_{123}$,

$$c'_{ij} \geq c_{ij} + \frac{2}{9}c_{123} \geq -\frac{2}{9}b > 0.$$

Similarly, by Corollary 2.3.4, for $(k,l)\neq(2,3)$,

$$c'_{1kl} = c_{1kl} + \frac{1}{9}c_{123} \geq 0.$$

□

The proofs of [18] and [21] each employ a different basis choice, and likewise involve moving the divisor within its linear equivalence class to find an effective representative. Via PORTA ([12]) and a short computer program written by the author, it can be shown that for the basis used above, there is precisely one extremal ray of the nef cone of $\overline{M}_{0,6}$ for each distinct $j,k\subseteq\{2,\ldots,6\}$ such that for the obvious divisor of Equation (2.3.5), $c_{1jk}<0$.

2.4 Fulton's conjecture for $\overline{M}_{0,7}$

The dimension of the Keel subspace $\mathcal{K}_7$ is 35, while $\dim N^1(\overline{M}_{0,7})_{\mathbb{Q}} = 42$, so an obvious basis candidate is to append divisor classes involving the seven psi-classes, plus correction terms involving B_2 and B_3. This choice can be used to prove Fulton's conjecture for $n=7$, but it does not generalize readily to larger n.

Instead, for $i,j=1,\ldots,7$, we define $D_i = \sum_{j\neq i}\Delta_{ij}$. The basis extension candidate is then

$$P_i = \alpha D_i + \lambda B_2 + \mu B_3, \tag{2.4.1}$$

where $i = 1, \ldots, 7$. We first look for α, λ, and μ that will extend the proof of Fulton's conjecture for $\mathcal{K}_7$ to all of $N^1(\overline{M}_{0,7})_{\mathbb{Q}}$. As with $\overline{M}_{0,6}$, this is possible for a codimension one subspace of $N^1(\overline{M}_{0,7})_{\mathbb{Q}}$, while outside of this subspace we use averages of Keel relations to find an effective representative.

Lemma 2.4.1. *The Keel classes plus the $[P_i]$, $i = 1, \ldots, 7$, form a basis for $N^1(\overline{M}_{0,7})_{\mathbb{Q}}$ if and only if $\alpha \neq 0$ and $18\alpha + 63\lambda - 35\mu \neq 0$.*

Proof. Corollary 2.2.9 states that the Keel classes plus the hyperplane class and the boundary classes $[\Delta_{i7}]$, $i = 1, \ldots, 6$, form a basis for $N^1(\overline{M}_{0,7})_{\mathbb{Q}}$, so to ensure that the $[P_i]$ complete the Keel classes to a basis, we calculate which α, λ, and μ guarantee that the projection map $\pi_{\mathcal{K}_7} : N^1(\overline{M}_{0,7})_{\mathbb{Q}} \to N^1(\overline{M}_{0,7})_{\mathbb{Q}}/\mathcal{K}_7$ restricted to the span of the $[P_i]$ give an isomorphism. Concretely, we write the $[P_i]$ in terms of the basis of Corollary 2.2.9, and check which values of α, λ, and μ correspond to a non-zero determinant for the base-change matrix. The class of D_7 is already written in terms of this basis; for $i = 1, \ldots, 6$ we propose the following form for $[D_i]$:

$$D_i = a \sum_{i,7\in I} S_I + b \sum_{\substack{i\in I \\ 7\notin I}} + c \sum_{\substack{i\notin I \\ 7\in I}} S_I + d \sum_{i,7\notin I} S_I + e\,\Delta_{i7} + f \sum_{\substack{j=1,\ldots,6 \\ j\neq i}} \Delta_{j7} + h\,H. \tag{2.4.2}$$

For concreteness, we determine $[D_1]$, the other $[D_i]$ being completely analogous. Intersections of all F-curves of $\overline{M}_{0,7}$ with the left and right sides of the proposed equality (2.4.2) for $i = 1$ are as follows, where, by a statement such as '1 on (2),' we mean that the marked point 1 is on a tail containing two marked points:

$$(*) \left\{ \begin{array}{rll} 2 = & a + 3\,b, & C \text{ type } (1:1:1:4), \text{ with 1 on spine, 7 on tail,} \\ 2 = & 4\,a + e + f + h, & C \text{ type } (1:1:1:4), \text{ with 1, 7 on spine,} \\ 0 = & a + 3\,c + 2\,f + h, & C \text{ type } (1:1:1:4), \text{ with 1 on tail, 7 on spine,} \\ 0 = & b + c + 2\,d, & C \text{ type } (1:1:1:4), \text{ with 1, 7 on tail,} \\ 1 = & 3\,a + 3\,b - f, & C \text{ type } (1:1:2:3), \text{ with 1 on spine, 7 on (2),} \\ 1 = & 2\,a + 4\,b, & C \text{ type } (1:1:2:3), \text{ with 1 on spine, 7 on (3),} \\ 1 = & 6\,a + e + h, & C \text{ type } (1:1:2:3), \text{ with 1, 7 on spine,} \\ -1 = & 3\,b + 3\,c - e, & C \text{ type } (1:1:2:3), \text{ with 1, 7 on (2),} \\ 0 = & 2\,b + 2\,c + 2\,d, & C \text{ type } (1:1:2:3), \text{ with 1, 7 on (3),} \\ -1 = & 3\,a + 3\,c + f + h, & C \text{ type } (1:1:2:3), \text{ with 1 on (2), 7 on spine,} \\ 0 = & 2\,a + 4\,c + f + h, & C \text{ type } (1:1:2:3), \text{ with 1 on (3), 7 on spine,} \\ 0 = & 4\,a + 4\,b - f, & C \text{ type } (1:2:2:2), \text{ with 1 on spine, 7 on tail,} \\ -1 = & 4\,a + 4\,c + h, & C \text{ type } (1:2:2:2), \text{ with 1 on tail, 7 on spine,} \\ -1 = & 4\,b + 4\,c - e, & C \text{ type } (1:2:2:2), \text{ with 1, 7 on same tail,} \\ -1 = & 2\,a + 2\,b + 2\,c & \\ & \quad +2\,d - f, & C \text{ type } (1:2:2:2), \text{ with 1, 7 on distinct tails.} \end{array} \right.$$

Gauss-Jordan elimination gives the (overdetermined but consistent) solution

$$[D_1] = -\frac{5}{2}\sum_{1,7\in I} + \frac{3}{2}\Big(\sum_{\substack{1\in I\\ 7\notin I}}[S_I] - \sum_{\substack{1\notin I\\ 7\in I}}[S_I]\Big) + [\Delta_{17}] - 4\sum_{j=2}^{6}[\Delta_{j7}] + 15[H].$$

To write $[B_2]$ and $[B_3]$ in terms of Keel classes, the $[\Delta_{i7}]$, $1 \leq i \leq 6$, and H, we may consider a simpler expression than that for D_i, and we need only differentiate F-curves according to where the marked point 7 is. Alternatively we may use the same expression and simply add two more columns to the right of the augmented matrix with the intersections of $[B_2]$ and $[B_3]$ with C, C', and C'', respectively $(1:1:1:4)$, $(1:1:2:3)$, and $(1:2:2:2)$ F-curves:

$$\begin{aligned} &[B_2]\cdot C = 3, &&[B_2]\cdot C' = 0, &&[B_2]\cdot C'' = -3,\\ &[B_3]\cdot C = -1, &&[B_3]\cdot C' = 1, &&[B_3]\cdot C'' = 3. \end{aligned}$$

The result is

$$\begin{aligned} [B_2] &= 3\sum_{7\notin I} S_I - 6\sum_{7\in I} S_I - 9\sum_{1\leq j\leq 6}[\Delta_{j7}] + 45H,\\ [B_3] &= -\frac{3}{2}\sum_{7\notin I} S_I + \frac{7}{2}\sum_{7\in I} S_I + 5\sum_{1\leq j\leq 6}[\Delta_{j7}] - 25H. \end{aligned}$$

Setting $x = -4\alpha - 9\lambda + 5\mu$, the matrix given by the coordinates of $[\Delta_{j7}]$, $1 \leq j \leq 6$ and H for the $[P_i]$ is

$$P = \begin{pmatrix} x+5\alpha & x & \dots & x & -5(x+\alpha)\\ x & x+5\alpha & & x & -5(x+\alpha)\\ \vdots & & \ddots & & \vdots\\ x & x & & x+5\alpha & -5(x+\alpha)\\ x+5\alpha & x+5\alpha & \dots & x+5\alpha & -5(x+4\alpha) \end{pmatrix}.$$

Row and column operations, followed by a Laplace expansion, give that the determinant is $-5(5\alpha)^6(-5(x+4\alpha)+6(\frac{3}{5}(x+5\alpha)))$, which, after substituting for x, yields the desired result, $\det(P) = -(5\alpha)^6(18\alpha + 63\lambda - 35\mu)$. ☐

Sufficient conditions for the determinant of Lemma 2.4.1 to vanish are easy to find. In particular, $\sum S_I = \frac{10}{3}\,B_2 + 6\,B_3$, while $\sum P_i = (2\,\alpha + 7\,\lambda)\,B_2 + 7\,\mu\,B_3$. These two sums are proportional if and only if $18\alpha + 63\lambda - 35\mu = 0$. Moreover, the choice $\alpha = 0$ obviously implies that the matrix is singular.

For $\alpha\,(18\alpha + 63\lambda - 35\mu) \neq 0$, every divisor class $[D]$ can be written uniquely as

$$[D] = \sum s_I[S_I] + \sum_{i=1}^{7} p_i[P_i]. \tag{2.4.3}$$

Taking the obvious representative, the coefficients of Δ_{12} and Δ_{123} in D are

$$
\begin{aligned}
c_{12} &= \frac{1}{3} \sum_{I \supseteq \{1,2\}} s_I + (\alpha + \lambda)(p_1 + p_2) + \lambda(p_3 + \ldots + p_7), \\
c_{123} &= \frac{1}{3} \sum_{|I \cap \{1,2,3\}| = 2} s_I + \mu(p_1 + \ldots + p_7).
\end{aligned}
$$

The cone of F-nef divisors is determined by the following three sets of inequalities, corresponding to partitions $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ of types $(1:1:1:4), (1:1:2:3)$, and $(1:2:2:2)$, respectively:

$$
\sum_{r=1}^{4} s_{I_r} + (2\alpha + 3\lambda - \mu) \sum_{i \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}} p_i + (3\lambda - \mu) \sum_{j \in \mathcal{D}} p_j \geq 0, \tag{2.4.4}
$$

$$
\sum_{r=1}^{6} s_{I'_r} + (\alpha + \mu) \sum_{i \in \mathcal{A} \cup \mathcal{B}} p_i + (-\alpha + \mu) \sum_{j \in \mathcal{C}} p_j + \mu \sum_{k \in \mathcal{D}} p_k \geq 0, \tag{2.4.5}
$$

$$
\sum_{r=1}^{8} s_{I''_r} + 3(-\lambda + \mu) \sum_{i \in \mathcal{A}} p_i + (-\alpha + 3(-\lambda + \mu)) \sum_{j \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}} p_j \geq 0, \tag{2.4.6}
$$

where all index sets are distinct within a given inequality, and, as in the notation of Lemma 2.2.2, satisfy $a_{I_r} = b_{I_r} = c_{I_r} = d_{I_r} = 1$ for all r (respectively $a_{I'_r} = \ldots = d_{I'_r} = 1$, $a_{I''_r} = \ldots = d_{I''_r} = 1$). For example, the inequality of type (2.4.5) for the curve $C(1, 2, \substack{3\\4}, \substack{5\\6\\7})$ is

$$
\begin{aligned}
&s_{1235} + s_{1236} + s_{1237} + s_{1245} + s_{1246} + s_{1247} \\
&+(\alpha + \mu)(p_1 + p_2) + (-\alpha + \mu)(p_3 + p_4) + \mu(p_5 + p_6 + p_7) \geq 0.
\end{aligned}
$$

Theorem 2.4.2 (Fulton's conjecture for $\overline{M}_{0,7}$)**.** *Let $[D]$ be an F-nef divisor class in $\overline{M}_{0,7}$. Then D is numerically equivalent to an effective sum of boundary divisors.*

Proof. By symmetry, the theorem is proven if we show that the coefficients c_{12} and c_{123} of the obvious representative D from Equation (2.4.3) are non-negative. The proof of Theorem 2.2.5 gives inequalities for c_{12} from $(1:1:1:4)$ and $(1:1:2:3)$ F-curves. We will combine these inequalities to conclude $c_{12} \geq 0$ without placing further restrictions on the parameters α, λ, and μ.

We consider first

$$
F_{12}^{(1:1:1:4)} = \{C(1,2,3,\substack{4\\5\\6\\7}), C(1,2,4,\substack{3\\5\\6\\7}), C(1,2,5,\substack{3\\4\\6\\7}), C(1,2,6,\substack{3\\4\\5\\7}), C(1,2,7,\substack{3\\4\\5\\6})\}.
$$

By the proof of Theorem 2.2.5 and Equation (2.4.4), we obtain that $D \cdot \sum_{C \in F_{12}^{(1:1:1:4)}} C$

equals

$$2\sum_{I\supseteq\{1,2\}} s_I + (10\alpha+15\lambda-5\mu)(p_1+p_2) + (2\alpha+15\lambda-5\mu)(p_3+\ldots+p_7).$$

Substituting $6c_{12}$ yields

$$6c_{12} + (4\alpha+9\lambda-5\mu)(p_1+p_2) + (2\alpha+9\lambda-5\mu)(p_3+\ldots+p_7) \geq 0.$$

Next, for

$$\begin{aligned} F_{12}^{(1:1:2:3)} = \{&C(1,2,\substack{3\\4},\substack{5\\6\\7}), C(1,2,\substack{3\\5},\substack{4\\6\\7}), C(1,2,\substack{3\\6},\substack{4\\5\\7}), C(1,2,\substack{3\\7},\substack{4\\5\\6}), C(1,2,\substack{4\\5},\substack{3\\6\\7}),\\ &C(1,2,\substack{4\\6},\substack{3\\5\\7}), C(1,2,\substack{4\\7},\substack{3\\5\\6}), C(1,2,\substack{5\\6},\substack{3\\4\\7}), C(1,2,\substack{5\\7},\substack{3\\4\\6}), C(1,2,\substack{6\\7},\substack{3\\4\\5})\}, \end{aligned}$$

we obtain as above

$$18c_{12} + (-8\alpha-18\lambda+10\mu)(p_1+p_2) + (-4\alpha-18\lambda+10\mu)(p_3+\ldots+p_7) \geq 0.$$

Combining, we see that for any allowed values of α, λ, and μ,

$$D\cdot\left(\frac{1}{15}\sum_{C\in F_{12}^{(1:1:1:4)}} C + \frac{1}{30}\sum_{C'\in F_{12}^{(1:1:2:3)}} C'\right) = c_{12} \geq 0.$$

Next we consider

$$\begin{aligned} F_{123}^{(1:1:2:3)} = \{&C(4,1,\substack{2\\3},\substack{5\\6\\7}), C(5,1,\substack{2\\3},\substack{4\\6\\7}), C(6,1,\substack{2\\3},\substack{4\\5\\7}), C(7,1,\substack{2\\3},\substack{4\\5\\6}), C(4,2,\substack{1\\3},\substack{5\\6\\7}), C(5,2,\substack{1\\3},\substack{4\\6\\7}),\\ &C(6,2,\substack{1\\3},\substack{4\\5\\7}), C(7,2,\substack{1\\3},\substack{4\\5\\6}), C(4,3,\substack{1\\2},\substack{5\\6\\7}), C(5,3,\substack{1\\2},\substack{4\\6\\7}), C(6,3,\substack{1\\2},\substack{4\\5\\7}), C(7,3,\substack{1\\2},\substack{4\\5\\6})\}, \end{aligned}$$

where we have reordered the partitions so that $\mathcal{B}\cup\mathcal{C} = \{1,2,3\}$. By the proof of Theorem 2.2.5 and Equation (2.4.5),

$$\begin{aligned} D\cdot\sum_{C\in F_{123}^{(1:1:2:3)}} C = 4&\sum_{|I\cap\{1,2,3\}|=2} s_I + (-4\alpha+12\mu)(p_1+p_2+p_3)\\ &+ (3\alpha+12\mu)(p_4+\ldots+p_7). \end{aligned}$$

Substituting for c_{123} gives $12c_{123} - 4\alpha(p_1+p_2+p_3) + 3\alpha(p_4+\ldots+p_7) \geq 0$.

Finally, for

$$\begin{aligned} F_{123}^{(1:2:2:2)} = \{&C(1,\substack{2\\3},\substack{4\\5},\substack{6\\7}), C(1,\substack{2\\3},\substack{4\\6},\substack{5\\7}), C(1,\substack{2\\3},\substack{4\\7},\substack{5\\6}), C(2,\substack{1\\3},\substack{4\\5},\substack{6\\7}), C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7}),\\ &C(2,\substack{1\\3},\substack{4\\7},\substack{5\\6}), C(3,\substack{1\\2},\substack{4\\5},\substack{6\\7}), C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7}), C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6})\}, \end{aligned}$$

we obtain $12c_{123} + (-6\alpha-27\lambda+15\mu)(p_1+p_2+p_3) + (-9\alpha-27\lambda+15\mu)(p_4+\ldots+p_7) \geq 0$.

Suppose next that $\alpha > 0$. The first inequality for c_{123} implies $c_{123} \geq 0$ provided that $\frac{4}{3}(p_1+p_2+p_3) \geq p_4+\ldots+p_7$. Supposing then that $p_4+\ldots+p_7 > \frac{4}{3}(p_1+p_2+p_3)$ and picking α, λ, and μ such that $9\alpha + 27\lambda - 15\mu \geq 0$, the second inequality for c_{123} implies

$$12c_{123} \geq (18\alpha + 63\lambda - 35\mu)(p_1 + p_2 + p_3),$$

so the conditions on α, λ, and μ that extend the proof from the Keel subspace ensure that the basis candidate under consideration does not yield a basis. Nevertheless, for $\alpha \neq 0$ and $18\alpha + 63\lambda - 35\mu = 0$, the rank of the intersection matrix P of Lemma 2.4.1 is six, thus proving the following proposition.

Proposition 2.4.3. *Let $[D]$ be an F-nef divisor class in the codimension one subpace $\langle S_I, P_i \rangle$, where $\alpha > 0$, $9\alpha + 27\lambda - 15\mu \geq 0$, and $18\alpha + 63\lambda - 35\mu = 0$. Then the obvious representative is an effective sum of boundary divisors.*

Outside of this codimension one subspace, the obvious representative of an F-nef divisor class written as in Equation (2.4.3) can have boundary terms with negative coefficients. To find an effective representative, we use Keel relations to obtain a new representative, and combine F-inequalities to prove that all boundary coefficients of this new representative are non-negative. Within the codimension one subspace of Proposition 2.4.3, we followed the proof of Fulton's conjecture for the Keel subspace (Theorem 2.2.5) to find suitable combinations of F-inequalities. Outside of this subspace, we use instead the simplex algorithm (see [15], or [56] for a succinct, geometrical account).

A principal use of the simplex algorithm is to minimize (or maximize) a linear functional subject to linear constraints. The algorithm terminates if the linear functional is expressed so that any move within the feasible region results in either no change, or an increase (or decrease) to the functional. By convexity, local minima (or maxima) give the global minimum (or maximum) of the functional.

In the present case, we take the coefficients of boundary divisors of the obvious representative D as the linear functionals, and use the simplex algorithm to obtain lower bounds subject to the F-inequalities. These bounds can then be verified by hand, since the coefficient in question is given at the last stage of the algorithm as a non-negative sum of the inequalities defining the feasible region. We use the implementation of the simplex algorithm in `lp_solve` ([5]) because it enables us to read off the linear functional from the final stage.

We may assume that c_{123} is the most negative of the boundary coefficients, and, by scaling, that $c_{123} = -1$. We fix a basis by setting $\alpha = 3, \lambda = 5$, and $\mu = 9$. Consider first the average of Keel relations for $-[\Delta_{123}]$ as in [21]:

$$\begin{aligned} -[\Delta_{123}] =& \frac{1}{3} \sum_{a,b \in \{1,2,3\}} [\Delta_{ab}] + \frac{1}{6} \sum_{x,y \in \{4,5,6,7\}} [\Delta_{xy}] - \frac{1}{6} \sum_{\substack{a \in \{1,2,3\}, \\ x,y \in \{4,5,6,7\}}} [\Delta_{axy}] \\ &+ \frac{1}{2} \sum_{x,y,z \in \{4,5,6,7\}} [\Delta_{xyz}] - \frac{1}{6} \sum_{\substack{a \in \{1,2,3\}, \\ x \in \{4,5,6,7\}}} [\Delta_{ax}] \end{aligned} \tag{2.4.7}$$

(we stipulate without further mention that indices in expressions as above are distinct and chosen to avoid double counting). Substituting the average of the obvious representative of $-\Delta_{123}$ from (2.4.7) into D yields an effective representative provided that the following four sets of inequalities are satisfied: (i) $c_{abx} \geq 0$, (ii) $c_{axy} \geq 1/6$, (iii) $c_{xyz} \geq -1/2$, and (iv) $c_{ax} \geq 1/6$ for all $a, b \in \{1,2,3\}$ and $x, y, z \in \{4,5,6,7\}$. To finish the proof, we consider each of these four collections of inequalities, and show that either a given inequality is satisfied, or that we can replace $-\Delta_{123}$ by a different average so that the resulting divisor is an effective sum of boundary. In most cases, we will actually find a sharper bound than is required.

For (i), we prove $c_{abx} \geq 3$ for all $a, b \in \{1,2,3\}$ and all $x \in \{4,5,6,7\}$. By symmetry, it suffices to prove $c_{124} \geq 3$. The last stage of the simplex algorithm gives c_{124} as the intersection of D with the following sum of F-curves:

$$\begin{aligned}
&\frac{1}{10}\big(C(1,2,5,\substack{3\\4\\6\\7}) + C(1,\substack{2\\3},\substack{4\\7},\substack{5\\6}) + C(2,\substack{1\\3},\substack{4\\5},\substack{6\\7}) + C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7}) + C(2,\substack{1\\3},\substack{4\\7},\substack{5\\6}) + C(5,\substack{1\\2},\substack{3\\4},\substack{6\\7})\big)\\
&+\frac{4}{15}C(2,5,\substack{1\\3},\substack{4\\6\\7}) + \frac{1}{30}\big(C(2,4,\substack{6\\7},\substack{1\\3\\5}) + C(2,5,\substack{3\\4},\substack{1\\6\\7}) + C(2,6,\substack{1\\3},\substack{4\\5\\7}) + C(2,7,\substack{1\\3},\substack{4\\5\\6})\big)\\
&+\frac{1}{15}\big(C(1,4,5,\substack{2\\3\\6\\7}) + C(2,4,\substack{1\\3},\substack{5\\6\\7})\big) + \frac{2}{15}\big(C(1,6,\substack{3\\4},\substack{2\\5\\7}) + C(1,7,\substack{3\\4},\substack{2\\5\\6}) + C(1,\substack{2\\3},\substack{4\\6},\substack{5\\7})\big)\\
&+\frac{1}{30}\big(C(1,4,6,\substack{2\\3\\5\\7}) + C(1,5,\substack{2\\3},\substack{4\\6\\7}) + C(1,6,\substack{2\\3},\substack{4\\5\\7}) + C(1,7,\substack{2\\3},\substack{4\\5\\6}) + C(1,7,\substack{5\\6},\substack{2\\3\\4})\big)\\
&+\frac{1}{5}\big(C(1,2,6,\substack{3\\4\\5\\7}) + C(1,2,7,\substack{3\\4\\5\\6}) + C(3,6,\substack{1\\2},\substack{4\\5\\7}) + C(3,7,\substack{1\\2},\substack{4\\5\\6}) + C(1,3,5,\substack{2\\4\\6\\7})\big)\\
&+\frac{1}{6}\big(C(2,6,\substack{5\\7},\substack{1\\3\\4}) + C(2,7,\substack{5\\6},\substack{1\\3\\4}) + C(1,\substack{2\\3},\substack{4\\5},\substack{6\\7})\big) + \frac{4}{5}C(3,4,\substack{1\\2},\substack{5\\6\\7}).
\end{aligned}$$

Next we intersect the above with D to obtain a linear inequality in the coefficients s_I and $p_1, \ldots, p_7$. For example, the coefficient of s_{1567} in the resulting inequality is 0, the coefficient of s_{1235} is $1/10 + 1/30 + 1/5 = 1/3$, the coefficient of s_{1256} is $1/10 + 1/10 + 1/10 + 4/15 + 1/30 + 2/15 + 1/30 + 1/30 + 1/5 + 1/6 + 1/6 = 4/3$, and the coefficient of p_1 is

$$\begin{aligned}
&\frac{1}{10}(12+12+12+9+9+9+9) + \frac{4}{15}(6)\\
&+\frac{1}{30}(9+9+6+6) + \frac{1}{15}(12+6)\\
&+\frac{2}{15}(12+12+12) + \frac{1}{30}(12+12+12+12+12)\\
&+\frac{1}{5}(12+12+6+6+12) + \frac{1}{6}(9+9+12) + \frac{4}{5}(6),
\end{aligned}$$

which equals 36.

The other coefficients are computed analogously, so that the intersection of the above

F-curve with D is

$$\begin{aligned}&\frac{1}{3}\sum_{\substack{|I\cap\{1,2,4\}|=2,\\|I\cap\{1,2,3\}|\neq 2}} s_I+\frac{4}{3}\sum_{\substack{|I\cap\{1,2,4\}|=2,\\|I\cap\{1,2,3\}|=2}} s_I+\sum_{\substack{|I\cap\{1,2,4\}|\neq 2,\\|I\cap\{1,2,3\}|=2}} s_I+36\sum_{i=1}^{7}p_i\\ =&c_{124}+3c_{123},\end{aligned}$$

so substituting $c_{123}=-1$ gives $c_{124}\geq 3$.

The remaining three sets of inequalities fail, so we must find different average expressions for $-[\Delta_{123}]$ in each case. In particular, we must now specify which of the Keel relations involving Δ_{123} to include in the average. Each four-tuple $(ijkl)$ determines three Keel relations, and Δ_{123} appears in these relations precisely when (up to reordering) $i,j\in\{1,2,3\}$ and $k,l\in\{4,5,6,7\}$. We will write the two relations involving Δ_{123} as $(ij)(kl)=(ik)(jl)$ and $(ij)(kl)=(il)(jk)$. To prove non-negativity of boundary coefficients for the new representative of D requires several sums of F-curves as with $c_{abx}\geq 3$, so we give only the resulting bounds below, and record the sums of F-curves giving these bounds in an appendix.

Case (ii)**:** $-1\leq c_{axy}\leq 1/6$**.** By symmetry, we consider only c_{145}. The Keel relation $(23)(45)=(24)(35)$ gives

$$\begin{aligned}-[\Delta_{123}]=&[\Delta_{236}]+[\Delta_{237}]+[\Delta_{145}]+[\Delta_{456}]+[\Delta_{457}]+[\Delta_{23}]+[\Delta_{45}]\\ &-([\Delta_{124}]+[\Delta_{246}]+[\Delta_{247}]+[\Delta_{135}]+[\Delta_{356}]+[\Delta_{357}])-([\Delta_{24}]+[\Delta_{35}]).\end{aligned}$$

Substituting the obvious representative of $-\Delta_{123}$ produces an effective representative of D provided no c_{ijk} besides c_{123} and c_{145} is negative, $c_{246}\geq 1$, and $c_{24}\geq 1$, since the other inequalities follow from symmetry.

We obtain $c_{246}\geq 17/3$ by substituting $-44/7\,c_{123}=44/7$ into the sum of the inequality $-26/7\,c_{145}\geq -26/42$ and the intersection of D with the sum of F-curves $C_{246}^{(ii)}$ of the appendix. The bound $c_{24}\geq 1$ follows by substituting $-8/7\,c_{123}=8/7$ into the sum of $-6/7\,c_{145}\geq -1/7$ and the intersection of D with the sum of F-curves $C_{24}^{(ii)}$ of the appendix.

It remains to show that no other c_{ijk} can be negative. By symmetry, the following inequalities (more than) suffice after after substituting $c_{123}=-1$:

(i) $c_{146}\geq 73/6$, which follows from $-65/7\,c_{145}+D\cdot C_{146}^{(ii)}\geq -65/42$;

(ii) $c_{167}\geq 83/6$, which follows from $-7\,c_{145}+D\cdot C_{167}^{(ii)}\geq -7/6$;

(iii) $c_{245}\geq 37/6$, which follows from $-29/7\,c_{145}+D\cdot C_{245}^{(ii)}\geq -29/42$;

(iv) $c_{267}\geq 59/6$, which follows from $-25/7\,c_{145}+D\cdot C_{267}^{(ii)}\geq -25/42$;

(v) $c_{456}\geq 9$, which follows from $-36/7\,c_{145}+D\cdot C_{456}^{(ii)}\geq -6/7$; and

(vi) $c_{467}\geq 13/2$, which follows from $-45/7\,c_{145}+D\cdot C_{467}^{(ii)}\geq -45/42$.

Case (iii)**:** $-1 \leq c_{xyz} \leq 0$**.** For Equation (2.4.7) to give an effective representative, we require only the upper bound $c_{456} \leq -1/2$, but considering $c_{456} \leq 0$ simplifies case (iv). Δ_{123} and Δ_{456} both appear in 18 Keel relations corresponding to the four-tuples (1245), (1246), (1256), (1345), (1346), (1356), (2345), (2346), and (2356), so we average over these relations, that is, we average over (12)(45) = (14)(25), (12)(45) = (15)(24), …, (23)(56) = (25)(36), (23)(56) = (26)(35) to obtain:

$$\begin{aligned}-[\Delta_{123}] =& [\Delta_{456}] - \frac{2}{9}\sum_{\substack{a\in\{1,2,3\},\\ x\in\{4,5,6\}}}([\Delta_{ax}]+[\Delta_{ax7}]) - \frac{1}{9}\Big(\sum_{\substack{a,b\in\{1,2,3\},\\ x\in\{4,5,6\}}}[\Delta_{abx}] + \sum_{\substack{a\in\{1,2,3\},\\ x,y\in\{4,5,6\}}}[\Delta_{axy}]\Big)\\ &+\frac{1}{3}\Big(\sum_{a,b\in\{1,2,3\}}[\Delta_{ab}] + \sum_{x,y\in\{4,5,6\}}[\Delta_{xy}] + \sum_{a,b\in\{1,2,3\}}[\Delta_{ab7}] + \sum_{x,y\in\{4,5,6\}}[\Delta_{xy7}]\Big).\end{aligned}$$

The following bounds show that substituting the obvious representatives from the above average for $-\Delta_{123}$ gives an effective representative of D:

(i) $c_{14} \geq 1$, which follows from $-1/3\, c_{456} + D \cdot C^{(iii)}_{14} \geq 0$;

(ii) $c_{145} \geq 13/3$, which follows from $-4/3\, c_{456} + D \cdot C^{(iii)}_{145} \geq 0$;

(iii) $c_{147} \geq 53/45$, which follows from $-7/45\, c_{456} + D \cdot C^{(iii)}_{147} \geq 0$; and

(iv) $c_{457} \geq 2$, which follows from $-c_{456} + D \cdot C^{(iii)}_{457} \geq 0$.

Case (iv)**:** $c_{ax} \leq 1/6$**.** We may now assume that $c_{axy} \geq 1/6$ and $c_{xyz} \geq 0$ for all $a \in \{1,2,3\}$ and $x,y,z \in \{4,5,6,7\}$. Suppose $c_{14} \leq 1/6$. We average over the 18 Keel relations involving Δ_{123} but not Δ_{14} corresponding to the four-tuples (1256), (1257), (1267), (1356), (1357), (1367), (2356), (2357), and (2367), that is, over the relations (12)(56) = (15)(26), (12)(56) = (16)(25), …, (23)(67) = (26)(37), (23)(67) = (27)(36):

$$\begin{aligned}-[\Delta_{123}] =& [\Delta_{567}] - \frac{2}{9}\sum_{\substack{a\in\{1,2,3\},\\ x\in\{5,6,7\}}}([\Delta_{ax}]+[\Delta_{a4x}]) - \frac{1}{9}\Big(\sum_{\substack{a,b\in\{1,2,3\},\\ x\in\{5,6,7\}}}[\Delta_{abx}] + \sum_{\substack{a\in\{1,2,3\},\\ x,y\in\{5,6,7\}}}[\Delta_{axy}]\Big)\\ &+\frac{1}{3}\Big(\sum_{a,b\in\{1,2,3\}}[\Delta_{ab}] + \sum_{x,y\in\{5,6,7\}}[\Delta_{xy}] + \sum_{a,b\in\{1,2,3\}}[\Delta_{ab4}] + \sum_{x,y\in\{5,6,7\}}[\Delta_{4xy}]\Big).\end{aligned}$$

The following bounds show that substituting the obvious representative for $-\Delta_{123}$ from the above average makes D an effective sum of boundary:

(i) $c_{15} \geq 7/6$, which follows from $-c_{14} + D \cdot C^{(iv)}_{15} \geq -1/6$;

(ii) $c_{145} \geq 257/198$, which follows from $-7/33\, c_{14} + D \cdot C^{(iv)}_{145} \geq -7/198$; and

(iii) $c_{245} \geq 41/36$, which follows from $-7/6\, c_{14} + D \cdot C^{(iv)}_{245} \geq -7/36$.

This completes the proof of Theorem 2.4.2. □

2.5 Appendix: F-curves used in the proof of Theorem 2.4.2

We record the sums of F-curves that give the inequalities on boundary coefficients used to finish the proof of Theorem 2.4.2.

Case (ii): $-1 \leq c_{145} \leq 1/6$.
For $c_{246} \geq 17/3$:

$$\begin{aligned}
C^{(ii)}_{246} = \quad & \tfrac{7}{30}\big(C(1,5,6,\substack{2\\3\\4\\7}) + C(2,\substack{1\\3},\substack{4\\7},\substack{5\\6}) + C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7})\big) + \tfrac{43}{70}C(1,3,\substack{4\\5},\substack{2\\6\\7}) \\
& +\tfrac{29}{70}C(2,3,5,\substack{1\\4\\6\\7}) + \tfrac{2}{105}C(2,3,6,\substack{1\\4\\5\\7}) + \tfrac{19}{42}\big((C(3,4,5,\substack{1\\2\\6\\7}) + C(2,4,\substack{6\\7},\substack{1\\3\\5})\big) \\
& +\tfrac{1}{3}\big((C(3,4,7,\substack{1\\2\\5\\6}) + C(2,7,\substack{4\\6},\substack{1\\3\\5}) + C(4,\substack{1\\5},\substack{2\\6},\substack{3\\7})\big) + \tfrac{22}{105}C(1,2,\substack{4\\5},\substack{3\\6\\7}) \\
& +\tfrac{46}{105}C(1,4,\substack{2\\3},\substack{5\\6\\7}) + \tfrac{4}{7}C(1,5,\substack{2\\3},\substack{4\\6\\7}) + \tfrac{58}{105}C(1,6,\substack{4\\5},\substack{2\\3\\7}) + \tfrac{9}{10}C(1,7,\substack{2\\3},\substack{4\\5\\6}) \\
& +\tfrac{1}{30}\big(C(2,4,\substack{1\\3},\substack{5\\6\\7}) + C(3,5,\substack{1\\2},\substack{4\\6\\7})\big) + \tfrac{11}{70}C(3,7,\substack{1\\2},\substack{4\\5\\6}) + \tfrac{17}{210}C(3,\substack{1\\2},\substack{4\\5},\substack{6\\7}) \\
& +\tfrac{24}{35}C(2,5,\substack{1\\3},\substack{4\\6\\7}) + \tfrac{33}{70}C(2,6,\substack{1\\3},\substack{4\\5\\7}) + \tfrac{13}{105}\big((C(2,7,\substack{1\\3},\substack{4\\5\\6}) + C(3,6,\substack{1\\2},\substack{4\\5\\7})\big) \\
& +\tfrac{1}{10}\big((C(4,5,\substack{2\\6},\substack{1\\3\\7}) + C(4,\substack{1\\5},\substack{2\\7},\substack{3\\6})\big) + \tfrac{10}{21}C(4,6,\substack{1\\5},\substack{2\\3\\7}) + \tfrac{23}{210}C(4,7,\substack{1\\5},\substack{2\\3\\6}) \\
& +\tfrac{11}{35}\big(C(1,3,6,\substack{2\\4\\5\\7}) + C(2,3,4,\substack{1\\5\\6\\7})\big) + \tfrac{22}{105}C(5,6,\substack{1\\4},\substack{2\\3\\7}) + \tfrac{51}{210}C(5,7,\substack{1\\4},\substack{2\\3\\6}) \\
& +\tfrac{11}{30}\big((C(1,\substack{2\\3},\substack{4\\5},\substack{6\\7}) + C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7}) + C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6})\big).
\end{aligned}$$

For $c_{24} \geq 1$:

$$\begin{aligned}
C^{(ii)}_{24} = \quad & \tfrac{4}{35}C(1,2,5,\substack{3\\4\\6\\7}) + \tfrac{5}{210}\big(C(1,3,5,\substack{2\\4\\6\\7}) + C(1,6,\substack{2\\3},\substack{4\\5\\7}) + C(3,5,\substack{1\\4},\substack{2\\6\\7})\big) \\
& +\tfrac{11}{210}C(3,4,5,\substack{1\\2\\6\\7}) + \tfrac{13}{210}\big(C(1,5,\substack{2\\3},\substack{4\\6\\7}) + C(3,6,\substack{1\\2},\substack{4\\5\\7}) + C(3,7,\substack{1\\2},\substack{4\\5\\6})\big) \\
& +\tfrac{37}{210}C(1,4,\substack{2\\3},\substack{5\\6\\7}) + \tfrac{1}{7}C(1,6,\substack{4\\5},\substack{2\\3\\7}) + \tfrac{1}{6}C(1,7,\substack{2\\3},\substack{4\\5\\6}) + \tfrac{13}{105}C(2,3,\substack{4\\5},\substack{1\\6\\7}) \\
& +\tfrac{31}{210}C(2,4,\substack{1\\5},\substack{3\\6\\7}) + \tfrac{1}{15}\big(C(2,4,\substack{3\\6},\substack{1\\5\\7}) + C(2,4,\substack{3\\7},\substack{1\\5\\6}) + C(3,5,\substack{6\\7},\substack{1\\2\\4})\big) \\
& +\tfrac{1}{210}C(2,4,\substack{5\\7},\substack{1\\3\\6}) + \tfrac{1}{105}C(2,4,\substack{6\\7},\substack{1\\3\\5}) + \tfrac{2}{21}\big(C(2,6,\substack{5\\7},\substack{1\\3\\4}) + C(2,7,\substack{1\\3},\substack{4\\5\\6})\big) \\
& +\tfrac{1}{14}\big(C(4,6,\substack{1\\5},\substack{2\\3}) + C(4,7,\substack{1\\5},\substack{2\\3\\6}) + C(5,6,\substack{1\\4},\substack{2\\3\\7}) + C(5,7,\substack{1\\4},\substack{2\\3\\6})\big) \\
& +\tfrac{11}{70}C(1,\substack{2\\3},\substack{4\\5},\substack{6\\7}) + \tfrac{1}{30}\big(C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7}) + C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6})\big) + \tfrac{3}{14}C(2,3,4,\substack{1\\5\\6\\7}) \\
& +\tfrac{16}{105}C(2,4,5,\substack{1\\3\\6\\7}) + \tfrac{11}{105}C(2,4,\substack{1\\3},\substack{5\\6\\7}) + \tfrac{1}{10}C(2,4,\substack{5\\6},\substack{1\\3\\7}).
\end{aligned}$$

For $c_{146} \geq 73/6$:

$$\begin{aligned}
C^{(ii)}_{146} = \quad & \tfrac{22}{15}C(1,2,5,\substack{3\\4\\6\\7}) + \tfrac{1}{14}C(1,3,5,\substack{2\\4\\6\\7}) + \tfrac{7}{30}C(1,3,6,\substack{2\\4\\5\\7}) + \tfrac{206}{105}C(2,3,4,\substack{1\\5\\6\\7}) \\
& + \tfrac{1}{3}\big(C(3,4,7,\substack{1\\2\\5\\6}) + C(4,5,7,\substack{1\\2\\3\\6}) + C(2,6,\substack{5\\7},\substack{1\\3\\4})\big) + \tfrac{242}{105}C(1,4,\substack{2\\3},\substack{5\\6\\7}) \\
& + \tfrac{12}{7}C(1,6,\substack{4\\5},\substack{2\\3\\7}) + \tfrac{1}{3}\big(C(3,5,\substack{1\\4},\substack{2\\6\\7}) + C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7}) + C(4,\substack{1\\5},\substack{2\\6},\substack{3\\7})\big) \\
& + \tfrac{1}{10}C(2,3,\substack{4\\6},\substack{1\\5\\7}) + \tfrac{46}{105}C(2,4,\substack{1\\5},\substack{3\\6\\7}) + \tfrac{12}{35}\big(C(2,7,\substack{1\\3},\substack{4\\5\\6}) + C(3,7,\substack{1\\2},\substack{4\\5\\6})\big) \\
& + \tfrac{11}{105}C(3,4,\substack{1\\5},\substack{2\\6\\7}) + \tfrac{223}{210}C(3,5,\substack{1\\2},\substack{4\\6\\7}) + \tfrac{16}{35}\big(C(4,6,\substack{1\\5},\substack{2\\3\\7}) + C(5,7,\substack{1\\4},\substack{2\\3\\6})\big) \\
& + \tfrac{44}{35}\big(C(5,6,\substack{1\\4},\substack{2\\3\\7}) + C(1,\substack{2\\3},\substack{4\\5},\substack{6\\7})\big) + \tfrac{12}{5}C(1,7,\substack{2\\3},\substack{4\\5\\6}) + \tfrac{13}{42}C(3,6,\substack{1\\2},\substack{4\\5\\7}) \\
& + \tfrac{2}{3}C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7}) + \tfrac{33}{35}C(5,\substack{1\\4},\substack{2\\3},\substack{6\\7}) + \tfrac{11}{15}C(2,3,5,\substack{1\\4\\6\\7}) + \tfrac{103}{210}C(2,3,\substack{4\\5},\substack{1\\6\\7}) \\
& + \tfrac{19}{35}C(2,6,\substack{1\\3},\substack{4\\5\\7}) + \tfrac{16}{105}C(1,6,\substack{2\\3},\substack{4\\5\\7}) + \tfrac{62}{105}C(4,7,\substack{1\\5},\substack{2\\3\\6}) \\
& + \tfrac{4}{5}\big(C(2,\substack{1\\3},\substack{4\\7},\substack{5\\6}) + C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6})\big).
\end{aligned}$$

For $c_{167} \geq 83/6$:

$$\begin{aligned}
C^{(ii)}_{167} = \quad & \tfrac{1}{3}\big(C(1,2,7,\substack{3\\4\\5\\6}) + C(1,4,7,\substack{2\\3\\5\\6}) + C(1,5,6,\substack{2\\3\\4\\7}) + C(2,3,6,\substack{1\\4\\5\\7})\big) \\
& + \tfrac{37}{30}C(1,3,5,\substack{2\\4\\6\\7}) + \tfrac{1}{3}\big(C(2,6,7,\substack{1\\3\\4\\5}) + C(3,4,\substack{5\\6},\substack{1\\2\\7}) + C(3,6,\substack{4\\7},\substack{1\\2\\5})\big) \\
& + \tfrac{29}{15}C(2,3,4,\substack{1\\5\\6\\7}) + \tfrac{19}{30}C(3,4,5,\substack{1\\2\\6\\7}) + \tfrac{19}{15}\big(C(1,4,\substack{2\\3},\substack{5\\6\\7}) + C(2,5,\substack{1\\3},\substack{4\\6\\7})\big) \\
& + \tfrac{7}{30}\big(C(1,3,\substack{4\\5},\substack{2\\6\\7}) + C(2,6,\substack{1\\3},\substack{4\\5\\7}) + C(2,7,\substack{1\\3},\substack{4\\5\\6}) + C(3,6,\substack{1\\2},\substack{4\\5\\7})\big) \\
& + \tfrac{5}{3}C(1,6,\substack{2\\3},\substack{4\\5\\7}) + \tfrac{2}{3}\big(C(1,7,\substack{2\\3},\substack{4\\5\\6}) + C(1,7,\substack{4\\5},\substack{2\\3\\6})\big) + \tfrac{1}{6}C(2,4,\substack{1\\5},\substack{3\\6\\7}) \\
& + \tfrac{1}{30}\big(C(2,5,\substack{1\\4},\substack{3\\6\\7}) + C(3,5,\substack{1\\4},\substack{2\\6\\7})\big) + \tfrac{12}{15}C(2,5,\substack{3\\4},\substack{1\\6\\7}) + \tfrac{1}{5}C(3,5,\substack{1\\2},\substack{4\\6\\7}) \\
& + \tfrac{5}{6}\big(C(4,6,\substack{1\\5},\substack{2\\3\\7}) + C(5,7,\substack{1\\4},\substack{2\\3\\6})\big) + \tfrac{1}{2}\big(C(4,7,\substack{1\\5},\substack{2\\3\\6}) + C(5,6,\substack{1\\4},\substack{2\\3\\7})\big) \\
& + \tfrac{2}{5}C(2,\substack{1\\3},\substack{4\\5},\substack{6\\7}) + \tfrac{23}{30}\big(C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7}) + C(2,\substack{1\\3},\substack{4\\7},\substack{5\\6}) + C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7})\big) \\
& + \tfrac{2}{15}C(1,5,\substack{2\\3},\substack{4\\6\\7}) + \tfrac{9}{10}C(3,7,\substack{1\\2},\substack{4\\5\\6}) + \tfrac{7}{3}C(1,\substack{2\\3},\substack{4\\5},\substack{6\\7}) + \tfrac{13}{30}C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6}).
\end{aligned}$$

For $c_{245} \geq 37/6$:

$$\begin{aligned}
C^{(ii)}_{245} = \quad & \tfrac{17}{14}C(1,2,4,\substack{3\\5\\6\\7}) + \tfrac{59}{70}C(2,3,5,\substack{1\\4\\6\\7}) + \tfrac{29}{210}C(1,4,\substack{2\\3},\substack{5\\6\\7}) + \tfrac{11}{21}C(1,5,\substack{2\\3},\substack{4\\6\\7}) \\
& + \tfrac{6}{7}C(1,6,\substack{4\\5},\substack{2\\3\\7}) + \tfrac{1}{105}\big(C(2,4,\substack{5\\7},\substack{1\\3\\6}) + C(2,7,\substack{4\\6},\substack{1\\3\\5}) + C(4,5,\substack{2\\6},\substack{1\\3\\7})\big) \\
& + \tfrac{18}{35}C(2,5,\substack{1\\4},\substack{3\\6\\7}) + \tfrac{8}{21}C(2,6,\substack{1\\3},\substack{4\\5\\7}) + \tfrac{34}{105}\big(C(2,7,\substack{5\\6},\substack{1\\3\\4}) + C(5,6,\substack{2\\4},\substack{1\\3\\7})\big) \\
& + \tfrac{7}{10}C(3,4,\substack{1\\2},\substack{5\\6\\7}) + \tfrac{3}{70}C(3,4,\substack{1\\5},\substack{2\\6\\7}) + \tfrac{16}{35}C(3,4,\substack{2\\5},\substack{1\\6\\7}) + \tfrac{5}{14}C(3,5,\substack{1\\4},\substack{2\\6\\7}) \\
& + \tfrac{6}{35}\big(C(3,6,\substack{1\\2},\substack{4\\5\\7}) + C(3,7,\substack{1\\2},\substack{4\\5\\6})\big) + \tfrac{1}{3}C(4,5,\substack{2\\7},\substack{1\\3\\6}) + \tfrac{76}{105}C(1,\substack{2\\3},\substack{4\\5},\substack{6\\7}) \\
& + \tfrac{11}{42}\big(C(4,6,\substack{1\\5},\substack{2\\3\\7}) + C(4,7,\substack{1\\5},\substack{2\\3\\6}) + C(5,6,\substack{1\\4},\substack{2\\3\\7}) + C(5,7,\substack{1\\4},\substack{2\\3\\6})\big) \\
& + \tfrac{1}{30}C(2,\substack{1\\3},\substack{4\\7},\substack{5\\6}) + \tfrac{8}{35}C(3,\substack{1\\2},\substack{4\\5},\substack{6\\7}) + \tfrac{2}{5}\big(C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7}) + C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6})\big) \\
& + \tfrac{1}{7}C(1,6,\substack{2\\3},\substack{4\\5\\7}) + C(1,7,\substack{2\\3},\substack{4\\5\\6}) + \tfrac{13}{35}C(2,7,\substack{1\\3},\substack{4\\5\\6}) \\
& + \tfrac{26}{105}C(2,\substack{1\\3},\substack{4\\5},\substack{6\\7}) + \tfrac{1}{42}C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7}).
\end{aligned}$$

For $c_{267} \geq 59/6$:

$$\begin{aligned}
C^{(ii)}_{267} = \quad & \tfrac{8}{21}\big(C(4,6,\substack{1\\5},\substack{2\\3\\7}) + C(4,7,\substack{1\\5},\substack{2\\3\\6}) + C(5,6,\substack{1\\4},\substack{2\\3\\7}) + C(5,7,\substack{1\\4},\substack{2\\3\\6}) + C(2,\substack{1\\3},\substack{4\\5},\substack{6\\7})\big) \\
& + \tfrac{104}{105}C(1,2,5,\substack{3\\4\\6\\7}) + \tfrac{1}{3}\big(C(2,3,6,\substack{1\\4\\5\\7}) + C(2,3,7,\substack{1\\5\\6\\7}) + C(3,6,7,\substack{1\\2\\4\\5})\big) \\
& + \tfrac{47}{35}C(1,4,\substack{2\\3},\substack{5\\6\\7}) + \tfrac{3}{2}C(1,6,\substack{2\\3},\substack{4\\5\\7}) + \tfrac{31}{42}C(1,7,\substack{2\\3},\substack{4\\5\\6}) + \tfrac{16}{21}C(1,7,\substack{4\\5},\substack{2\\3\\6}) \\
& + \tfrac{1}{10}C(2,4,\substack{6\\7},\substack{1\\3\\5}) + \tfrac{16}{105}\big(C(2,6,\substack{1\\3},\substack{4\\5\\7}) + C(2,7,\substack{1\\3},\substack{4\\5\\6}) + C(5,\substack{1\\4},\substack{2\\3},\substack{6\\7})\big) \\
& + \tfrac{1}{30}C(3,4,\substack{1\\2},\substack{5\\6\\7}) + \tfrac{43}{42}C(3,5,\substack{1\\2},\substack{4\\6\\7}) + \tfrac{1}{5}C(3,5,\substack{1\\4},\substack{2\\6\\7}) + \tfrac{34}{105}C(3,\substack{1\\2},\substack{4\\5},\substack{6\\7}) \\
& + \tfrac{5}{42}\big(C(3,6,\substack{1\\2},\substack{4\\5\\7}) + C(3,7,\substack{1\\2},\substack{4\\5\\6})\big) + \tfrac{1}{2}\big(C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7}) + C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6})\big) \\
& + \tfrac{7}{15}\big(C(1,\substack{2\\3},\substack{4\\5},\substack{6\\7}) + C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7}) + C(2,\substack{1\\3},\substack{4\\7},\substack{5\\6})\big) + \tfrac{25}{21}C(2,3,4,\substack{1\\5\\6\\7}) \\
& + \tfrac{8}{105}C(3,4,5,\substack{1\\2\\6\\7}) + \tfrac{22}{105}C(2,4,\substack{1\\3},\substack{5\\6\\7}) + \tfrac{107}{210}C(2,5,\substack{6\\7},\substack{1\\3\\4}).
\end{aligned}$$

For $c_{456} \geq 9$:

$$\begin{aligned} C^{(ii)}_{456} = {} & \tfrac{1}{30}\big(C(2,4,\substack{1\\3},\substack{5\\6\\7}) + C(2,5,\substack{1\\4},\substack{3\\6\\7}) + C(3,4,\substack{1\\5},\substack{2\\6\\7}) + C(4,5,\substack{2\\7},\substack{1\\3\\6}) + C(4,5,\substack{3\\7},\substack{1\\2\\6})\big) \\ & + \tfrac{29}{70}\big(C(1,2,5,\substack{3\\4\\6\\7}) + C(5,7,\substack{1\\4},\substack{2\\3\\6})\big) + \tfrac{11}{210}C(1,4,7,\substack{2\\3\\5\\6}) + \tfrac{137}{105}C(1,6,7,\substack{2\\3\\4\\5}) \\ & + \tfrac{121}{105}C(2,3,5,\substack{1\\4\\6\\7}) + \tfrac{5}{42}C(2,6,7,\substack{1\\3\\4\\5}) + \tfrac{82}{105}C(1,4,\substack{2\\3},\substack{5\\6\\7}) + \tfrac{19}{14}C(1,5,\substack{2\\3},\substack{4\\6\\7}) \\ & + \tfrac{3}{70}C(1,7,\substack{2\\3},\substack{4\\5\\6}) + \tfrac{47}{105}C(2,4,\substack{1\\5},\substack{3\\6\\7}) + \tfrac{3}{14}\big(C(2,4,\substack{5\\6},\substack{1\\3\\7}) + C(5,7,\substack{2\\4},\substack{1\\3\\6})\big) \\ & + \tfrac{2}{7}C(2,7,\substack{1\\3},\substack{4\\5\\6}) + \tfrac{1}{3}\big(C(3,4,\substack{5\\6},\substack{1\\2\\7}) + C(3,5,\substack{4\\7},\substack{1\\2\\6})\big) + \tfrac{4}{35}C(3,5,\substack{2\\4},\substack{1\\6\\7}) \\ & + \tfrac{17}{42}C(3,7,\substack{1\\2},\substack{4\\5\\6}) + \tfrac{23}{70}\big(C(4,6,\substack{1\\5},\substack{2\\3\\7}) + C(5,6,\substack{1\\4},\substack{2\\3\\7}) + C(3,\substack{1\\2},\substack{4\\5},\substack{6\\7})\big) \\ & + \tfrac{38}{105}C(4,7,\substack{1\\5},\substack{2\\3\\6}) + \tfrac{41}{21}C(1,\substack{2\\3},\substack{4\\5},\substack{6\\7}) + \tfrac{1}{2}\big(C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7}) + C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7})\big) \\ & + \tfrac{2}{7}C(2,\substack{1\\3},\substack{4\\7},\substack{5\\6}) + \tfrac{1}{6}C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6}) + \tfrac{19}{35}C(4,\substack{1\\5},\substack{2\\3},\substack{6\\7}) + \tfrac{37}{30}C(2,3,4,\substack{1\\5\\6\\7}) \\ & + \tfrac{31}{105}C(1,6,\substack{2\\3},\substack{4\\5\\7}) + \tfrac{19}{42}C(2,6,\substack{1\\3},\substack{4\\5\\7}) + \tfrac{4}{7}C(3,6,\substack{1\\2},\substack{4\\5\\7}). \end{aligned}$$

For $c_{467} \geq 13/2$:

$$\begin{aligned} C^{(ii)}_{467} = {} & \tfrac{199}{420}C(1,2,4,\substack{3\\5\\6\\7}) + \tfrac{541}{420}C(1,2,5,\substack{3\\4\\6\\7}) + \tfrac{10}{7}C(1,6,7,\substack{2\\3\\4\\5}) + \tfrac{27}{35}C(2,3,4,\substack{1\\5\\6\\7}) \\ & + \tfrac{7}{60}\big(C(4,5,7,\substack{1\\2\\3\\6}) + C(3,4,\substack{1\\5},\substack{2\\6\\7}) + C(5,6,\substack{3\\7},\substack{1\\2\\4})\big) + \tfrac{23}{84}C(1,3,\substack{4\\5},\substack{2\\6\\7}) \\ & + \tfrac{2}{7}\big(C(1,2,\substack{4\\5},\substack{3\\6\\7}) + C(4,6,\substack{1\\5},\substack{2\\3\\7}) + C(4,7,\substack{1\\5},\substack{2\\3\\6})\big) + \tfrac{1}{30}C(2,4,\substack{1\\3},\substack{5\\6\\7}) \\ & + \tfrac{1}{84}C(2,5,\substack{3\\4},\substack{1\\6\\7}) + \tfrac{13}{28}C(2,6,\substack{1\\3},\substack{4\\5\\7}) + \tfrac{209}{420}C(2,7,\substack{1\\3},\substack{4\\5\\6}) + \tfrac{131}{210}C(3,4,\substack{1\\2},\substack{5\\6\\7}) \\ & + \tfrac{13}{60}\big(C(2,7,\substack{4\\6},\substack{1\\3\\5}) + C(4,5,\substack{2\\7},\substack{1\\3\\7})\big) + \tfrac{13}{10}C(3,5,\substack{1\\2},\substack{4\\6\\7}) + \tfrac{26}{105}C(3,6,\substack{1\\2},\substack{4\\5\\7}) \\ & + \tfrac{139}{420}C(3,7,\substack{1\\2},\substack{4\\5\\6}) + \tfrac{1}{4}\big(C(4,5,\substack{2\\6},\substack{1\\3\\7}) + C(2,\substack{1\\3},\substack{4\\7},\substack{5\\6})\big) + \tfrac{267}{140}C(1,\substack{2\\3},\substack{4\\5},\substack{6\\7}) \\ & + \tfrac{53}{210}\big(C(5,6,\substack{1\\4},\substack{2\\3\\7}) + C(5,7,\substack{1\\4},\substack{2\\3\\6})\big) + \tfrac{23}{60}C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7}) + \tfrac{2}{15}C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6}) \\ & + \tfrac{141}{140}C(4,\substack{1\\5},\substack{2\\3},\substack{6\\7}) + \tfrac{4}{35}C(5,\substack{1\\4},\substack{2\\3},\substack{6\\7}) + \tfrac{17}{60}C(5,\substack{1\\4},\substack{2\\7},\substack{3\\6}) + \tfrac{1}{3}C(3,6,\substack{4\\7},\substack{1\\2\\5}) \\ & + \tfrac{1}{14}\big(C(1,6,\substack{4\\5},\substack{2\\3\\7}) + C(1,7,\substack{4\\5},\substack{2\\3\\6})\big) + \tfrac{31}{140}C(2,4,\substack{1\\5},\substack{3\\6\\7}) + \tfrac{1}{12}C(4,5,\substack{3\\6},\substack{1\\2\\7}). \end{aligned}$$

Case (iii)**:** $-1 \leq c_{456} \leq 0$**.**

For $c_{14} \geq 1$:

$$
\begin{aligned}
C_{14}^{(iii)} = \;& \tfrac{1}{9}\big(C(1,4,5,\substack{2\\3\\6\\7}) + C(1,6,\substack{4\\7},\substack{2\\3\\5})\big) + \tfrac{4}{45}\big(C(1,4,6,\substack{2\\3\\5\\7}) + C(1,5,\substack{2\\3},\substack{4\\6\\7})\big) \\
& + \tfrac{1}{18}\big(C(1,3,4,\substack{2\\5\\6\\7}) + C(2,5,7,\substack{1\\3\\4\\6}) + C(2,6,7,\substack{1\\3\\4\\5})\big) + \tfrac{1}{5}C(1,2,4,\substack{3\\5\\6\\7}) \\
& + \tfrac{1}{18}\big(C(3,5,\substack{4\\6},\substack{1\\2\\7}) + C(4,7,\substack{5\\6},\substack{1\\2\\3}) + C(1,\substack{2\\3},\substack{4\\6},\substack{5\\7})\big) + \tfrac{2}{15}C(1,4,7,\substack{2\\3\\5\\6}) \\
& + \tfrac{1}{30}C(3,6,7,\substack{1\\2\\4\\5}) + \tfrac{7}{90}\big(C(1,4,\substack{2\\7},\substack{3\\5\\6}) + C(1,\substack{2\\3},\substack{4\\5},\substack{6\\7}) + C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6})\big) \\
& + \tfrac{2}{45}\big(C(1,4,\substack{2\\3},\substack{5\\6\\7}) + C(2,5,\substack{1\\3},\substack{4\\6\\7}) + C(2,6,\substack{1\\3},\substack{4\\5\\7})\big) + \tfrac{13}{90}C(3,4,\substack{1\\2},\substack{5\\6\\7}) \\
& + \tfrac{1}{90}C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7}) + \tfrac{2}{45}\big(C(3,5,\substack{1\\2},\substack{4\\6\\7}) + C(3,6,\substack{1\\2},\substack{4\\5\\7}) + C(3,\substack{1\\2},\substack{4\\5},\substack{6\\7})\big) \\
& + \tfrac{1}{15}\big(C(1,4,\substack{5\\6},\substack{2\\3\\7}) + C(2,\substack{1\\3},\substack{4\\5},\substack{6\\7}) + C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7})\big) \\
& + \tfrac{1}{45}\big(C(1,4,\substack{5\\7},\substack{2\\3\\6}) + C(3,6,\substack{4\\5},\substack{1\\2\\7})\big).
\end{aligned}
$$

For $c_{145} \geq 13/3$:

$$
\begin{aligned}
C_{145}^{(iii)} = \;& \tfrac{1}{3}\big(C(1,2,4,\substack{3\\5\\6\\7}) + C(1,2,5,\substack{3\\4\\6\\7}) + C(2,5,\substack{1\\3},\substack{4\\6\\7})\big) + \tfrac{17}{45}C(3,5,\substack{1\\2},\substack{4\\6\\7}) \\
& + \tfrac{11}{45}C(3,5,6,\substack{1\\2\\4\\7}) + \tfrac{13}{90}\big(C(1,5,\substack{4\\6},\substack{2\\3\\7}) + C(2,6,\substack{4\\5},\substack{1\\2\\7}) + C(1,\substack{2\\3},\substack{4\\7},\substack{5\\6})\big) \\
& + \tfrac{5}{9}C(1,6,\substack{2\\3},\substack{4\\5\\7}) + \tfrac{1}{9}C(1,6,\substack{4\\5},\substack{2\\3\\7}) + \tfrac{47}{90}C(1,7,\substack{4\\5},\substack{2\\3\\6}) + \tfrac{7}{15}C(3,4,\substack{1\\2},\substack{5\\6\\7}) \\
& + \tfrac{1}{45}\big(C(2,4,\substack{1\\3},\substack{5\\6\\7}) + C(4,\substack{1\\3},\substack{2\\7},\substack{5\\6}) + C(7,\substack{1\\5},\substack{2\\3},\substack{4\\6})\big) + \tfrac{1}{30}C(1,4,7,\substack{2\\3\\5\\6}) \\
& + \tfrac{4}{45}\big(C(2,4,\substack{6\\7},\substack{1\\3\\5}) + C(3,5,\substack{6\\7},\substack{1\\2\\4}) + C(4,7,\substack{1\\5},\substack{2\\3\\6})\big) + \tfrac{1}{10}C(5,7,\substack{3\\4},\substack{1\\2\\6}) \\
& + \tfrac{7}{90}\big(C(2,6,\substack{1\\3},\substack{4\\5\\7}) + C(4,\substack{1\\7},\substack{2\\3},\substack{5\\6}) + C(5,\substack{1\\7},\substack{2\\3},\substack{4\\6})\big) + \tfrac{3}{10}C(3,4,\substack{5\\6},\substack{1\\2\\7}) \\
& + \tfrac{11}{90}C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6}) + \tfrac{1}{10}C(4,5,7,\substack{1\\2\\3\\6}) + \tfrac{1}{5}\big(C(2,6,\substack{4\\7},\substack{1\\3\\5}) + C(3,7,\substack{1\\2},\substack{4\\5\\6})\big) \\
& + \tfrac{2}{45}\big(C(2,4,6,\substack{1\\3\\5\\7}) + C(4,7,\substack{3\\5},\substack{1\\2\\6})\big) + \tfrac{8}{45}\big(C(2,4,\substack{5\\6},\substack{1\\3\\7}) + C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7})\big) \\
& + \tfrac{2}{9}\big(C(2,7,\substack{1\\3},\substack{4\\5\\6}) + C(3,6,\substack{1\\2},\substack{4\\5\\7}) + C(1,\substack{2\\3},\substack{4\\5},\substack{6\\7}) + C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7})\big).
\end{aligned}
$$

For $c_{147} \geq 53/45$:

$$
\begin{aligned}
C^{(iii)}_{147} = \;& \tfrac{31}{450}C(1,2,4,\substack{3\\5\\6\\7}) + \tfrac{4}{135}C(1,3,4,\substack{2\\5\\6\\7}) + \tfrac{2}{9}C(1,5,6,\substack{2\\3\\4\\7}) + \tfrac{221}{1350}C(2,4,7,\substack{1\\3\\5\\6}) \\
&+\tfrac{28}{675}\big(C(4,5,7,\substack{1\\2\\3\\6}) + C(2,6,\substack{4\\5},\substack{1\\3\\7})\big) + \tfrac{7}{135}C(4,6,7,\substack{1\\2\\3\\5}) + \tfrac{281}{1350}C(1,2,\substack{4\\7},\substack{3\\5\\6}) \\
&+\tfrac{59}{1350}\big(C(2,6,\substack{1\\3},\substack{4\\5\\7}) + C(3,6,\substack{1\\2},\substack{4\\5\\7})\big) + \tfrac{23}{1350}C(1,7,\substack{2\\3},\substack{4\\5\\6}) + \tfrac{139}{675}C(3,4,\substack{1\\2},\substack{5\\6\\7}) \\
&+\tfrac{47}{1350}C(3,7,\substack{1\\4},\substack{2\\5\\6}) + \tfrac{7}{150}C(3,7,\substack{4\\6},\substack{1\\2\\5}) + \tfrac{2}{45}\big(C(4,5,\substack{1\\7},\substack{2\\3\\6}) + C(4,6,\substack{1\\7},\substack{2\\3\\5})\big) \\
&+\tfrac{7}{1350}\big(C(4,7,\substack{2\\5},\substack{1\\3\\6}) + C(5,7,\substack{3\\4},\substack{1\\2\\6})\big) + \tfrac{7}{675}C(5,7,\substack{3\\6},\substack{1\\2\\4}) + \tfrac{451}{1350}C(1,\substack{2\\3},\substack{4\\7},\substack{5\\6}) \\
&+\tfrac{53}{1350}\big(C(5,7,\substack{1\\4},\substack{2\\3\\6}) + C(6,7,\substack{1\\4},\substack{2\\3\\5})\big) + \tfrac{91}{1350}C(3,\substack{1\\2},\substack{4\\5},\substack{6\\7}) + \tfrac{7}{135}C(7,\substack{1\\4},\substack{2\\6},\substack{3\\5}) \\
&+\tfrac{1}{1350}C(2,4,\substack{3\\7},\substack{1\\5\\6}) + \tfrac{23}{270}C(3,5,\substack{1\\2},\substack{4\\6\\7}) + \tfrac{82}{675}C(3,7,\substack{5\\6},\substack{1\\2\\4}) + \tfrac{49}{675}C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7}) \\
&+\tfrac{1}{90}\big(C(1,5,\substack{4\\7},\substack{2\\3\\6}) + C(1,6,\substack{4\\7},\substack{2\\3\\5})\big) + \tfrac{26}{675}\big(C(2,5,\substack{1\\3},\substack{4\\6\\7}) + C(2,7,\substack{3\\4},\substack{1\\5\\6})\big) \\
&+\tfrac{7}{270}\big(C(3,4,\substack{1\\7},\substack{2\\5\\6}) + C(2,\substack{1\\3},\substack{4\\5},\substack{6\\7}) + C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7})\big) + \tfrac{7}{450}C(3,4,\substack{5\\6},\substack{1\\2\\7}).
\end{aligned}
$$

For $c_{457} \geq 2$:

$$
\begin{aligned}
C^{(iii)}_{457} = \;& \tfrac{1}{10}\big(C(1,\substack{2\\3},\substack{4\\6},\substack{5\\7}) + C(1,\substack{2\\3},\substack{4\\7},\substack{5\\6}) + C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7}) + C(2,\substack{1\\3},\substack{4\\7},\substack{5\\6}) + C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7})\big) \\
&+\tfrac{1}{15}\big(C(1,4,6,\substack{2\\3\\7}) + C(2,5,6,\substack{1\\3\\4\\7}) + C(2,5,7,\substack{1\\3\\4\\6}) + C(1,4,\substack{2\\3},\substack{5\\6\\7})\big) \\
&+\tfrac{1}{15}\big(C(1,5,\substack{4\\6},\substack{2\\3\\7}) + C(1,7,\substack{4\\5},\substack{2\\3\\6}) + C(2,4,\substack{1\\3},\substack{5\\6\\7}) + C(2,4,\substack{5\\6},\substack{1\\3\\7})\big) \\
&+\tfrac{1}{15}\big(C(2,4,\substack{5\\7},\substack{1\\3\\6}) + C(2,5,\substack{1\\3},\substack{4\\6\\7}) + C(3,4,\substack{1\\2},\substack{5\\6\\7}) + C(3,4,\substack{5\\6},\substack{1\\2\\7})\big) \\
&+\tfrac{1}{15}\big(C(3,5,\substack{1\\2},\substack{4\\6\\7}) + C(3,\substack{1\\2},\substack{4\\5},\substack{6\\7}) + C(6,\substack{1\\2},\substack{3\\7},\substack{4\\5})\big) + \tfrac{1}{30}C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6}) \\
&+\tfrac{2}{15}\big(C(1,6,7,\substack{2\\3\\4\\5}) + C(2,6,7,\substack{1\\3\\4\\5})\big) + \tfrac{1}{5}\big(C(2,4,5,\substack{1\\3\\6\\7}) + C(3,6,\substack{1\\2},\substack{4\\5\\7})\big) \\
&+\tfrac{4}{15}\big(C(3,4,7,\substack{1\\2\\5\\6}) + C(1,4,\substack{6\\7},\substack{2\\3\\5}) + C(2,\substack{1\\3},\substack{4\\5},\substack{6\\7})\big) \\
&+\tfrac{1}{3}\big(C(1,5,\substack{2\\3},\substack{4\\6\\7}) + C(3,5,\substack{4\\7},\substack{1\\2\\6}) + C(6,7,\substack{4\\5},\substack{1\\2\\3})\big).
\end{aligned}
$$

Case *(iv)***:** $c_{14} \leq 1/6$**.**

For $c_{15} \geq 7/6$:

$$\begin{aligned}
C_{15}^{(iv)} = {} & \tfrac{1}{15}\big(C(1,5,6,\substack{2\\3\\4\\7}) + C(1,4,\substack{5\\6},\substack{2\\3\\7}) + C(1,7,\substack{4\\5},\substack{2\\3\\6}) + C(2,6,\substack{1\\3},\substack{4\\5\\7}) + C(2,7,\substack{1\\3},\substack{4\\5\\6})\\
& + \tfrac{1}{45}\big(C(1,4,\substack{2\\5},\substack{3\\6\\7}) + C(1,4,\substack{3\\5},\substack{2\\6\\7})\big) + \tfrac{1}{15}\big(C(3,6,\substack{1\\2},\substack{4\\5\\7}) + C(3,7,\substack{1\\2},\substack{4\\5\\6})\big)\\
& + \tfrac{1}{6}C(1,2,4,\substack{3\\5\\6\\7}) + \tfrac{17}{90}\big(C(1,2,5,\substack{3\\4\\6\\7}) + C(1,6,7,\substack{2\\3\\4\\5})\big) + \tfrac{1}{30}C(1,3,4,\substack{2\\5\\6\\7})\\
& + \tfrac{7}{45}C(1,4,5,\substack{2\\3\\6\\7}) + \tfrac{7}{18}C(1,4,\substack{6\\7},\substack{2\\3\\5}) + \tfrac{6}{15}C(1,5,\substack{2\\3},\substack{4\\6\\7}) + \tfrac{4}{45}C(2,3,\substack{4\\5},\substack{1\\6\\7})\\
& + \tfrac{1}{90}\big(C(1,4,\substack{2\\3},\substack{5\\6\\7}) + C(1,6,\substack{5\\7},\substack{2\\3\\4}) + C(1,7,\substack{4\\6},\substack{2\\3\\5}) + C(1,\substack{2\\3},\substack{4\\7},\substack{5\\6})\big)\\
& + \tfrac{2}{45}\big(C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7}) + C(2,\substack{1\\3},\substack{4\\7},\substack{5\\6}) + C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7}) + C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6})\big)\\
& + \tfrac{2}{15}\big(C(3,4,\substack{1\\2},\substack{5\\6\\7}) + C(3,5,\substack{1\\2},\substack{4\\6\\7})\big) + \tfrac{1}{18}C(1,3,5,\substack{2\\4\\6\\7}).
\end{aligned}$$

For $c_{145} \geq 257/198$:

$$\begin{aligned}
C_{145}^{(iv)} = {} & \tfrac{7}{99}\big(C(4,5,7,\substack{1\\2\\3\\6}) + C(2,\substack{1\\3},\substack{4\\6},\substack{5\\7}) + C(2,\substack{1\\3},\substack{4\\7},\substack{5\\6}) + C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7}) + C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6})\big)\\
& + \tfrac{49}{330}C(1,2,5,\substack{3\\4\\6\\7}) + \tfrac{23}{330}C(1,3,4,\substack{2\\5\\6\\7}) + \tfrac{4}{99}\big(C(3,6,\substack{1\\2},\substack{4\\5\\7}) + C(3,7,\substack{1\\2},\substack{4\\5\\6})\big)\\
& + \tfrac{5}{198}C(1,5,6,\substack{2\\3\\4\\7}) + \tfrac{4}{45}C(2,3,5,\substack{1\\4\\6\\7}) + \tfrac{31}{495}C(2,4,\substack{1\\5},\substack{3\\6\\7}) + \tfrac{46}{495}C(3,5,\substack{1\\2},\substack{4\\6\\7})\\
& + \tfrac{26}{495}C(2,4,5,\substack{1\\3\\6\\7}) + \tfrac{1}{22}C(4,5,6,\substack{1\\2\\3\\7}) + \tfrac{2}{9}C(1,6,\substack{2\\3},\substack{4\\5\\7}) + \tfrac{49}{198}C(1,7,\substack{4\\5},\substack{2\\3\\6})\\
& + \tfrac{1}{90}C(2,4,\substack{3\\5},\substack{1\\6\\7}) + \tfrac{53}{495}C(3,4,\substack{1\\2},\substack{5\\6\\7}) + \tfrac{7}{990}C(3,4,\substack{1\\5},\substack{2\\6\\7}) + \tfrac{103}{990}C(3,4,\substack{2\\5},\substack{1\\6\\7})\\
& + \tfrac{1}{18}C(3,5,\substack{1\\4},\substack{2\\6\\7}) + \tfrac{38}{495}C(4,6,\substack{1\\5},\substack{2\\3\\7}) + \tfrac{17}{330}C(4,7,\substack{1\\5},\substack{2\\3\\6}) + \tfrac{119}{495}C(1,\substack{2\\3},\substack{4\\5},\substack{6\\7})\\
& + \tfrac{1}{33}\big(C(5,6,\substack{1\\4},\substack{2\\3\\7}) + C(5,7,\substack{1\\4},\substack{2\\3\\6})\big) + \tfrac{7}{495}\big(C(4,\substack{1\\5},\substack{2\\6},\substack{3\\7}) + C(4,\substack{1\\5},\substack{2\\7},\substack{3\\6})\big)\\
& + \tfrac{4}{99}\big(C(2,5,\substack{3\\4},\substack{1\\6\\7}) + C(2,6,\substack{1\\3},\substack{4\\5\\7}) + C(2,7,\substack{1\\3},\substack{4\\5\\6})\big) + \tfrac{4}{33}C(1,2,4,\substack{3\\5\\6\\7}).
\end{aligned}$$

For $c_{245} \geq 41/36$:

$$
\begin{aligned}
C^{(iv)}_{245} = \ & \tfrac{7}{180}\big(C(1,4,\substack{2\\3},\substack{5\\6\\7}) + C(1,4,\substack{6\\7},\substack{2\\3\\5}) + C(3,5,\substack{1\\2},\substack{4\\6\\7}) + C(3,\substack{1\\2},\substack{4\\6},\substack{5\\7}) + C(3,\substack{1\\2},\substack{4\\7},\substack{5\\6})\big) \\
& + \tfrac{8}{45}C(1,2,4,\substack{3\\5\\6\\7}) + \tfrac{1}{9}\big(C(1,4,6,\substack{2\\3\\5\\7}) + C(1,\substack{2\\3},\substack{4\\6},\substack{5\\7}) + C(1,\substack{2\\3},\substack{4\\7},\substack{5\\6})\big) \\
& + \tfrac{17}{90}C(1,4,7,\substack{2\\3\\5\\6}) + \tfrac{1}{6}\big(C(3,4,5,\substack{1\\2\\6\\7}) + C(2,6,\substack{4\\5},\substack{1\\3\\7})\big) + \tfrac{5}{18}C(1,2,\substack{4\\5},\substack{3\\6\\7}) \\
& + \tfrac{7}{90}\big(C(1,4,\substack{3\\7},\substack{2\\5\\6}) + C(4,\substack{1\\7},\substack{2\\5},\substack{3\\6})\big) + \tfrac{1}{18}C(2,3,\substack{4\\5},\substack{1\\6\\7}) + \tfrac{11}{180}C(2,5,\substack{1\\3},\substack{4\\6\\7}) \\
& + \tfrac{1}{180}\big(C(1,4,\substack{5\\6},\substack{2\\3\\7}) + C(1,4,\substack{5\\7},\substack{2\\3\\6}) + C(4,7,\substack{2\\5},\substack{1\\3\\6})\big) + \tfrac{11}{60}C(3,\substack{1\\2},\substack{4\\5},\substack{6\\7}) \\
& + \tfrac{7}{36}C(2,7,\substack{1\\3},\substack{4\\5\\6}) + \tfrac{1}{10}C(3,4,\substack{1\\2},\substack{5\\6\\7}) + \tfrac{13}{180}\big(C(3,6,\substack{1\\2},\substack{4\\5\\7}) + C(3,7,\substack{1\\2},\substack{4\\5\\6})\big) \\
& + \tfrac{1}{90}C(1,4,\substack{3\\5},\substack{2\\6\\7}) + \tfrac{1}{20}C(2,\substack{1\\3},\substack{4\\5},\substack{6\\7}) + \tfrac{1}{36}C(2,6,\substack{1\\3},\substack{4\\5\\7}) + \tfrac{2}{45}C(1,3,4,\substack{2\\5\\6\\7}) \\
& + \tfrac{14}{45}C(1,4,5,\substack{2\\3\\6\\7}) + \tfrac{1}{12}\big(C(4,6,\substack{2\\5},\substack{1\\3\\7}) + C(5,6,\substack{2\\4},\substack{1\\3\\7}) + C(5,7,\substack{2\\4},\substack{1\\3\\6})\big).
\end{aligned}
$$

3 On relations in the Cox ring of $\overline{M}_{0,n}$

3.1 Introduction

Cox rings were introduced in [14] as a generalization of the homogeneous coordinate ring of projective space to toric varieties. For a toric variety X, Cox established a quotient construction of X analogous to the construction of projective space as $\mathbb{C}^{n+1} \setminus 0$ by $\mathbb{C}^*$. Hu and Keel in [34] further generalized Cox's construction to a broad class of projective varieties (see Definition 3.2.9), and proved far-reaching implications for the birational geometry of the variety when its Cox ring is finitely generated. They showed that the projective variety can be recovered as a quotient of the spectrum of its Cox ring by the action of its Picard torus, and that the cones of pseudoffective, nef, and moving divisor classes have a particularly nice structure. Specifically, Propositions 2.9 and 1.11 of [34] guarantee that, for a projective variety X with a finitely generated Cox ring:

(i) The pseudoeffective cone $\overline{\mathrm{Eff}}(X)$ is finitely generated, and there are finitely many birational contractions $g_i : X \dashrightarrow Y_i$, where Y_i is also projective with finitely generated Cox ring, such that $\overline{\mathrm{Eff}}(X)$ admits a decomposition

$$\overline{\mathrm{Eff}}(X) = \bigcup_i \left(g_i^*(\mathrm{Nef}(Y_i)\right) \times \mathrm{ex}(g_i),$$

where $\mathrm{ex}(g_i) \subseteq \overline{\mathrm{Eff}}(X)$ is locus of exceptional effective divisors of g_i. These chambers are closed, convex cones with disjoint interiors.

(ii) There exists a finite number of *small* $\mathbb{Q}$*-factorial modifications* $f_i : X \dashrightarrow X_i$ (that is, f_i is a contracting birational map, with X_i a $\mathbb{Q}$-factorial, normal, projective variety, and f_i an isomorphism in codimension one) such that the cone of moving divisor classes has a decomposition,

$$\mathrm{Mov}(X) = \bigcup_i f_i^*(\mathrm{Nef}(X_i)).$$

Furthermore, the f_i give all of the small $\mathbb{Q}$-factorial modifications of X, and the chambers $f_i^*(\mathrm{Nef}(X_i))$, along with their faces, endow $\mathrm{Mov}(X)$ with a fan structure.

(iii) The nef cone $\mathrm{Nef}(X)$ is the affine hull of finitely many semi-ample line bundles.

Via the above and additional consequences of finite-generation of the Cox ring of X, Hu and Keel proved that such varieties are in a sense ideally suited for the minimal model program, and hence they named these varieties *Mori dream spaces*.

Proving finite-generation of Cox rings of various varieties has gained significant attention. One of the many consequences of [6] is that any log-Fano variety is a Mori dream space. Turning to the moduli space of stable pointed rational curves, it follows that for $n \leq 6$, the Cox ring of $\overline{M}_{0,n}$ is finitely generated, since $\overline{M}_{0,6}$ is log-Fano, while for $n = 4$ and 5, $\overline{M}_{0,n}$ is a Fano variety. Moreover, explicit generators for the Cox ring of $\overline{M}_{0,6}$ were determined in [10].

A very elementary consequence of finite-generation is that the Cox ring admits a presentation as the quotient of a polynomial ring, with the variables corresponding to the generators. Even though finite-generation implies the existence of the above-mentioned characterizations of the pseudoeffective, moving, and nef cones of divisor classes, it is first necessary to determine the ideal of relations among the generators to calculate these cones and their decompositions.

In this chapter, we establish two main results about the ideal of relations of the Cox ring of $\overline{M}_{0,n}$. To describe these results, we establish some initial notation, but postpone full definitions until Section 3.2. Let $\mathcal{G}_{bnd} = \{x_J : \Delta_J \text{ is a boundary divisor}\}$ be the variables in sections of all boundary divisors, and let I_{bnd} be the kernel of the homomorphism $\mathbb{C}[\mathcal{G}_{bnd}] \to \mathrm{Cox}(\overline{M}_{0,n})$, i.e. I_{bnd} is the ideal of relations among boundary section variables.

The first result establishes collections of subrings of $\mathbb{C}[\mathcal{G}_{bnd}]$ that inject into $\mathrm{Cox}(\overline{M}_{0,n})$ in the nicest possible way: the variable $x_J \in \mathbb{C}[\mathcal{G}_{bnd}]$ is sent to the same variable in $\mathrm{Cox}(\overline{M}_{0,n})$.

Theorem 3.1.1. *Let $i, j \in \{1, \ldots, n\}$ be distinct, and let $\mathcal{G}_{bnd}(i,j) \subseteq \mathcal{G}_{bnd}$ be*

$$\mathcal{G}_{bnd}(i,j) = \{x_J : \Delta_J \textit{ a boundary divisor, } |J \cap \{i,j\}| = 1\}.$$

Then there is an multigraded endomorphism $\mathbb{C}[\mathcal{G}_{bnd}(i,j)] \hookrightarrow \mathrm{Cox}(\overline{M}_{0,n})$ defined by $x_J \to x_J$. In particular, $I_{bnd} \cap \mathbb{C}[\mathcal{G}_{bnd}(i,j)] = 0$.

We find these relation-free boundary section variables by studying the relation between the Losev-Manin moduli space $\overline{L}_{n-2}$ and $\overline{M}_{0,n}$. In the original Kapranov blow-up construction of $\overline{M}_{0,n}$, [36], these spaces serve as an intermediate between projective space $\mathbb{P}^{n-3}$ and $\overline{M}_{0,n}$. Since the Losev-Manin moduli spaces are toric varieties, their Cox rings are polynomial (see [14]), with each variable corresponding to a torus-invariant divisor. We obtain the relation-free collections by showing that $\mathrm{Cox}(\overline{L}_{n-2})$ injects as a graded $\mathbb{C}$-algebra into $\mathrm{Cox}(\overline{M}_{0,n})$ by sending a boundary section variable for $\overline{L}_{n-2}$ to a naturally corresponding variable for $\overline{M}_{0,n}$. Establishing this correspondence is that main task of Section 3.3. That these this morphism is an injection follows from a basic property of pull-backs of proper surjective morphisms (see Lemma 3.3.15).

To find relations among boundary generators in the Cox ring of $\overline{M}_{0,n}$, we consider divisor classes that $[F_{J,m}]$ that induce a composition of the forgetful morphism $\pi_J : \overline{M}_{0,n} \to \overline{M}_{0,n-|J|}$, followed by the Kapranov morphism $t_m : \overline{M}_{0,n-|J|} \to \mathbb{P}^{n-|J|-3}$.

Theorem 3.1.2. *Let $J \subseteq \{1, \ldots, n\}$, with $1 \leq |J| \leq n-4$, and let $m \in J^c$. Then for $n \geq 6$, the graded part $\mathbb{C}[\mathcal{G}_{bnd}]_{[F_{J,m}]}$ corresponding to the class $[F_{J,m}]$ interesects I_{bnd} nontrivially, with* $\dim(I_{bnd} \cap \mathbb{C}[\mathcal{G}_{bnd}]_{[F_{J,m}]}) = (n-|J|-2)(n-|J|-3)/2$.

Thus the first, 'relation-free' Theorem gives $\binom{n}{2}$ non-trivial polynomial subrings of $\mathrm{Cox}(\overline{M}_{0,n})$, while the second, 'relation-full' theorem gives subrings isomorphic to non-trivial quotients of polynomial rings. For $n = 5$, the relations described in Theorem 3.1.2 generate all relations in $\mathrm{Cox}(\overline{M}_{0,5})$ (see [4] or [43]). The Cox ring of $\overline{M}_{0,6}$ is finitely generated, and progress has been made towards finding generators besides sections of boundary and Keel-Vermeire divisors for $n \geq 7$ in [11], but the ideal of relations has not been studied for $n \geq 6$.

The paper is organized as follows. In Section 3.2, we introduce the Losev-Manin moduli spaces $\overline{L}_{n-2}$, present basic facts and notation from toric geometry used in the paper, and then define the Cox ring of a projective variety. Section 3.3 contains the proof of Theorem 3.1.1, obtained by studying the blow-up constructions of $\overline{L}_{n-2}$ and $\overline{M}_{0,n}$, with attention focused on how divisor classes in $\overline{L}_{n-2}$ pull back to $\overline{M}_{0,n}$. Of particular use is the language of 'clean intersections.' Originally defined by Bott in [7] in the context of differential geometry, we use the more algebraic formulation of [47]. Section 3.4 then contains the proof of Theorem 3.1.2.

3.2 Background and notation

We begin by giving a brief account of the Losev-Manin moduli space $\overline{L}_{n-2}$, which in many ways parallels $\overline{M}_{0,n}$, as described in Section 1.2. We refer to [48] and [3] for further details. The spaces $\overline{L}_{n-2}$ compactify projective equivalence classes of $n-2$ ordered points on $\mathbb{P}^1 - \{0, \infty\}$. In the case of $\mathbb{P}^1$, the points 0 and ∞ are called *poles.* The compactification results by adding in reducible pointed curves as with $\overline{M}_{0,n}$

Definition 3.2.1. A *stable (n-2)-pointed chain of projective lines* is a projective curve $C = C_1 \cup \ldots \cup C_m$ with marked points $p_1, \ldots, p_{n-2} \in C$ and poles $p_- \in C_1$, $p_+ \in C_m$ such that

(i) each C_j is isomorphic to $\mathbb{P}^1$,

(ii) a non-trivial intersection of two components is a simple double point,

(iii) two components C_i and C_j intersect if and only if $|i - j| = 1$,

(iv) marked points are distinct from singular points and from the poles p_-, p_+,

(v) each component contains at least one marked point.

Note that the stability condition for $\overline{L}_{n-2}$ is the same as that for $\overline{M}_{0,n}$ once we widen the definition of a 'special point' to include the two poles. Figure 3.1 shows two elements of $\overline{L}_3$.

A key feature of Losev-Manin moduli spaces is that they are toric varieties described as a composition of blow-ups of projective space along torus-invariant linear centers. It will be useful in the proof of Theorem 3.1.1 to describe these blow-ups explicitly in terms of fans and cones, so we now turn attention to toric blow-ups.

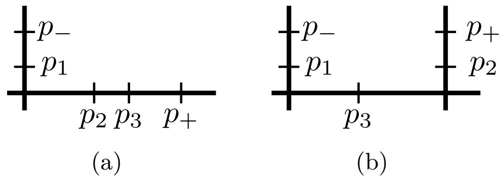


Figure 3.1: Elements of $\overline{L}_3$

Our sources for toric geometry are [13] and [22]. Let N be a free abelian group of rank d with dual lattice M, and let X_Σ be a toric variety of dimension d with fan $\Sigma \subseteq N \otimes_{\mathbb{Z}} \mathbb{R} = N_{\mathbb{R}}$ and torus T_N. We denote the set of k-dimensional cones of Σ by $\Sigma(k)$. A central fact for what follows is the Orbit-Cone correspondence.

Theorem 3.2.2. *There exists a bijective correspondence between cones $\sigma \in \Sigma$ and T_N-orbits $O(\sigma)$ such that if $\sigma \in \Sigma(k)$, then* $\dim(O(\sigma)) = d - k$.

Of particular importance are the closures of these orbits. For $\sigma \in \Sigma(k)$, let N_σ be the sublattice of N generated by points of $\sigma \cap N$.

Proposition 3.2.3. *For $\sigma \in \Sigma(k)$, the orbit closure $V(\sigma) = \overline{O(\sigma)}$ is the toric variety $X_{\mathrm{Star}(\sigma)}$, where*

$$\mathrm{Star}(\sigma) = \{\overline{\tau} : \sigma \textit{ is a face of } \tau\},$$

and $\overline{\tau}$ is the image of τ under the projection map $N_{\mathbb{R}} \to (N/N_\sigma)_{\mathbb{R}}$.

In anticipation of a correspondence with the stratification of $\overline{M}_{0,n}$ described in Section 1.2.3 of Chapter 1, we make the following definition.

Definition 3.2.4. The *codimension k-strata* of the toric variety X_Σ are the subvarieties $V(\sigma)$, where $\sigma \in \Sigma(k)$. The codimension one-strata of X_Σ are called *boundary divisors*, and will be denoted $D_\rho = V(\rho)$, where $\rho \in \Sigma(1)$.

The reason for labeling these divisors as 'boundary' is that same as for $\overline{M}_{0,n}$: for $\overline{M}_{0,n}$, the union of the boundary divisors equals the complement $\overline{M}_{0,n} \setminus M_{0,n}$, while in the toric case, the union of the boundary divisors equals $X_\Sigma \setminus T_N$. Note further that the strata of the toric variety X_Σ share two of the nice properties of the strata of $\overline{M}_{0,n}$: the union of the codimension one strata is a normal crossing divisor, and each codimension k stratum is a complete intersection of k boundary divisors (see [22], Section 5.1).

We consider one of the simplest—and for us the most relevant—examples of the Orbit-Cone correspondence, that of the complex projective space $\mathbb{P}^d$. Later we will concentrate on blow-ups of $\mathbb{P}^d$ along linear subspaces, so first we introduce some notation.

Notation 3.2.5 (Linear subspaces of $\mathbb{P}^d$)**.** For nonempty $J \subseteq \{1, \ldots, d+1\}$, let $l_J \subseteq \mathbb{P}^d$ be the coordinate subspace

$$l_J = \{[x_1, \ldots, x_{d+1}] \in \mathbb{P}^d : x_i = 0 \text{ if } i \in J^c\}.$$

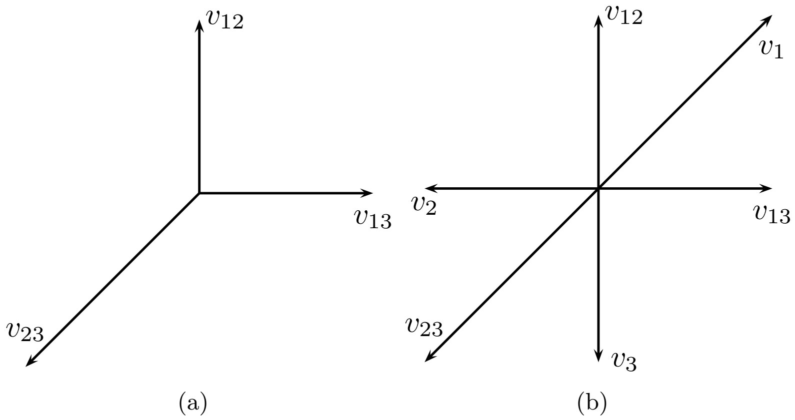


Figure 3.2: Rays of fans of $\mathbb{P}^2$ and $\overline{L}_3$

Example 3.2.6. Let $e_1, \ldots, e_d$ be standard basis vectors for the vector space $\mathbb{C}^d$, and define $f_1 = -e_1 - e_2 - \ldots - e_d$, $f_2 = e_1$, $\ldots$, $f_{d+1} = e_d$. The generating vectors for the rays of the fan of $\mathbb{P}^d$ are $f_1, \ldots, f_{d+1}$, but we will relabel them to highlight Notation 3.2.5 and the Orbit-Cone correspondence. We index generating vectors v_J by subsets $J \subseteq \{1, \ldots, d+1\}$, $|J| = d$, as follows: $v_{1\ldots d-1\,d} = f_{d+1}$, $v_{1\ldots d-1\,d+1} = f_d$, $\ldots$, $v_{2\ldots d\,d+1} = f_1$.

In our notation the coordinate hyperplanes are written as l_J for $J \subseteq \{1, \ldots, d+1\}$, $|J| = d$, and the Orbit-Cone correspondence translates into the equality $l_J = V(\langle v_J \rangle_{\geq 0})$ for such J.

For a general linear subspace l_J, the Orbit-Cone correspondence dictates that l_J is the orbit closure of the cone $\tau \in \Sigma(d + 1 - |J|)$ generated by vectors $v_{J'}$ satisfying $J' \supseteq J$, that is,

$$l_J = V(\langle v_{J'} : J' \supseteq J \rangle_{\geq 0}).$$

Specializing to $\mathbb{P}^2$, the rays of the fan of $\mathbb{P}^2$ are depicted in Figure 3.2 (a). The coordinate hyperplanes are

$$\begin{aligned}
l_{12} &= \{[\lambda, \mu, 0] : [\lambda, \mu] \in \mathbb{P}^1\} = V(\langle v_{12} \rangle_{\geq 0}) = V(\langle (0,1)^t \rangle_{\geq 0}),\\
l_{13} &= \{[\lambda, 0, \mu] : [\lambda, \mu] \in \mathbb{P}^1\} = V(\langle v_{13} \rangle_{\geq 0}) = V(\langle (1,0)^t \rangle_{\geq 0}),\\
l_{23} &= \{[0, \lambda, \mu] : [\lambda, \mu] \in \mathbb{P}^1\} = V(\langle v_{23} \rangle_{\geq 0}) = V(\langle (-1,-1)^t \rangle_{\geq 0}),
\end{aligned}$$

and the torus-invariant points are

$$\begin{aligned}
l_1 &= [1, 0, 0] = V(\langle v_{12}, v_{13} \rangle_{\geq 0},\\
l_2 &= [0, 1, 0] = V(\langle v_{12}, v_{23} \rangle_{\geq 0},\\
l_3 &= [0, 0, 1] = V(\langle v_{13}, v_{23} \rangle_{\geq 0}.
\end{aligned}$$

We next describe the toric interpretation of a blow-up along a torus-invariant center.

Let $\Sigma \subseteq N$ be a d-dimensional fan, and let $\sigma = \langle u_1, \ldots, u_d \rangle_{\geq 0}$ be a smooth cone (that is, $u_1, \ldots, u_d$ form a $\mathbb{Z}$-basis for the lattice N). To construct the blow-up of X_Σ along $V(\sigma)$, let $u = u_1 + \ldots + u_d$, and define σ' to be the set of all cones generated by subsets of $\{u, u_1, \ldots, u_d\}$ that do not contain $\{u_1, \ldots, u_d\}$. Then the fan of the blow-up $Bl_{V(\sigma)}(X_\Sigma)$ is

$$\Sigma' = (\Sigma \setminus \sigma) \cup \sigma'.$$

The exceptional divisor of the blow-up is D_u, and the proper transform of the divisor D_{u_i}, $i = 1, \ldots, d$, is $D_{u_i} - D_u$ (for ρ not a face of σ, the proper transform leaves D_ρ unchanged).

Notation 3.2.7 (Blow-ups and proper transforms)**.** We will label blow-ups along a coordinate subspace by the index of the center being blown-up. For example, $f_J : X_J \to \mathbb{P}^d$ will denote the blow-up of $\mathbb{P}^d$ along l_J. For proper transforms of linear subspaces under iterated blow-ups, we will in general abuse notation by not demarcating the proper transform, but rather indicating which proper transform is intended via the ambient variety. For example, we will write $l_J \subseteq X_{J'}$ for the proper transform of l_J under the blow-up $f_{J'}$ (and all blow-ups preceding $f_{J'}$). An exception to this (abuse of) notation is when the focus is on how a subvariety behaves under proper transform (as, for example, in the proof of Proposition 3.3.6). In such cases, we denote the proper transform of a subvariety $V \subseteq X$ under a blow-up $f_J : X_J \to X$ by $\widetilde{V}$.

Cox rings were first defined for toric varieties in [14] (see also Chapter 5 of [13]). The Cox (or total coordinate) ring of the toric variety X_Σ is the polynomial ring

$$\mathrm{Cox}(X_\Sigma) = \mathbb{C}[x_\rho : \rho \in \Sigma(1)].$$

This ring has a $\mathrm{Pic}(X_\Sigma)$-grading defined by

$$\deg\Big(\prod_{\rho \in \Sigma(1)} x_\rho^{a_\rho}\Big) = \Big[\sum_{\rho \in \Sigma(1)} a_\rho D_\rho\Big].$$

For $\alpha \in \mathrm{Pic}(X_\Sigma)$, we label the α-graded part of the Cox ring by $\mathrm{Cox}(X_\Sigma)_\alpha$. If a divisor $D = \sum a_\rho D_\rho$ has class α, there exists a non-canonical isomorphism $H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) \to \mathrm{Cox}(X_\Sigma)_\alpha$. To obtain this isomorphism, for such a divisor D we define a polytope $P_D \subseteq N_\mathbb{R}^\vee = M_\mathbb{R}$ by

$$P_D = \{m \in M : \langle m, u_\rho \rangle \geq -a_\rho\}. \tag{3.2.1}$$

The integral points of P_D then give a basis $\{\chi^m : m \in P_D\}$ of $H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$. The isomorphism between $H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ and $\mathrm{Cox}(X_\Sigma)_\alpha$ is defined by the map

$$\chi^m \mapsto \prod_{\rho \in \Sigma(1)} x_\rho^{\langle m, u_\rho \rangle + a_\rho}$$

(see [13], Sections 4.3 and 5.4, for more details and proofs). A consequence of the above

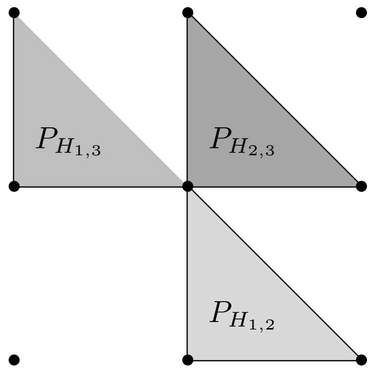


Figure 3.3: Polytopes of the hyperplane class in $\overline{L}_3$

isomorphism is that the dimension of $H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ can be calculated by counting integral points of the polytope P_D.

Thus the monomials in $\mathrm{Cox}(X_\Sigma)$ correspond to global sections of divisors classes on X_Σ. To form a 'global section ring' isomorphic to the Cox ring, however, we need to account for different representative divisors for a given divisor class. We consider an example involving $\mathbb{P}^2$ blown up three points. In the next section, we will identify this blow-up as the Losev-Manin space $\overline{L}_3$.

Example 3.2.8. By the above discussion of toric blow-ups, the fan of $\mathbb{P}^2$ blown up at $[1,0,0]$, $[0,1,0]$, and $[0,0,1]$ is as depicted in Figure 3.2 (b), with the $D_i = V(\langle v_i \rangle_{\geq 0})$ the exceptional divisors, and the $D_{ij} = V(\langle v_{ij} \rangle_{\geq 0})$ the proper transforms of the lines $(x_k = 0)$, where $\{i,j,k\} = \{1,2,3\}$.

The pull-backs of the three torus-invariant lines can be written as an integral sum of exceptional divisors as $H_{1,2} = D_1 + D_2 + D_{12}$, $H_{1,3} = D_1 + D_3 + D_{13}$, and $H_{2,3} = D_2 + D_3 + D_{23}$. The respective polytopes, defined as in Equation (3.2.1), are depicted in Figure 3.3.

For each of the three polytopes, the three integral points map to the monomials $x_1\, x_2\, x_{12}$, $x_1\, x_3\, x_{13}$, and $x_2\, x_3\, x_{23}$.

The absence of a canonical identification between the α-graded part of $\mathrm{Cox}(X_\Sigma)$ and global sections of a divisor whose class is α can be remedied by selecting divisors $D_1, \ldots, D_r$ whose classes form a basis for $\mathrm{Pic}(X_\Sigma)$. With this choice, multiplication of sections is induced by multiplication of functions in $\mathbb{C}(X_\Sigma)$, bringing us to the more general definition of the Cox ring of a projective variety, which was introduced in [34].

Definition 3.2.9. Let X be a projective variety with a torsion-free Picard group satisfying $\mathrm{Pic}(X)_\mathbb{Q} = N^1(X)_\mathbb{Q}$. Let $D_1, \ldots, D_r$ be divisors whose classes form a basis of $\mathrm{Pic}(X)_\mathbb{Q}$. The *Cox ring* of X with respect to this choice of divisors is

$$\mathrm{Cox}(X) = \sum_{(m_1,\ldots,m_r)\in\mathbb{Z}^r} H^0(X, \mathcal{O}_X(m_1 D_1 + \ldots + m_r D_r)),$$

with multiplication given by multiplication of functions in $\mathbb{C}(X)$.

It is proved in [34] that different choices of divisors yield non-canonically isomorphic Cox rings.

Before turning to the proof of Theorem 3.1.1, we make one further remark about notation. In contrast to the other chapters, we refer to the Picard group, $\mathrm{Pic}(X)$ (or the vector space $\mathrm{Pic}(X)_{\mathbb{Q}}$) rather than the Néron-Severi space, $N^1(X)_{\mathbb{Q}}$, but we maintain the notation $[D] \in \mathrm{Pic}(X)$ to signify the class of the divisor, i.e. the isomorphism class of the line bundle $\mathcal{O}_X(D)$.

3.3 Relation free generators in the Cox ring of $\overline{M}_{0,n}$

In [36], Kapranov constructed $\overline{M}_{0,n}$ as a composition of blow-ups of $\mathbb{P}^{n-3}$ by first considering a closely related toric variety, called a *permutohedral space* since its fan can be defined via the permutohedron, a polytope whose convex hull is defined by permutations of the vector $(1, 2, \ldots, d) \in \mathbb{R}^d$. This toric variety has also been studied by Procesi ([52]), and later given a modular interpretation in [48]. Since we will be focused on the modular interpretation and its connection to $\overline{M}_{0,n}$, we will simply refer to these toric varieites as the Losev-Manin moduli spaces $\overline{L}_{n-2}$.

To construct $\overline{L}_{n-2}$ as a blow-up of $\mathbb{P}^{n-3}$, we make use of the notational conventions 3.2.5 and 3.2.7. We first blow up $l_1 = [1, 0, \ldots, 0]$, then the proper transform $l_2 \in X_1$, where $f_1 : X_1 \to \mathbb{P}^{n-3}$ is the blow-up along l_1, continuing until we have blown-up $l_{n-2} \in X_{n-3}$. Next we blow up the proper transform of the line $l_{12} \subseteq X_{n-2}$, then the line $l_{13} \subseteq X_{12}$, continuing until all proper transforms of lines are blown up. We proceed in this way, blowing up proper transforms of coordinate subspaces in increasing order of dimension until all proper transforms of codimension two coordinate subspaces have been blown up. Note that this ordering respects the partial ordering by inclusion on the linear subspaces whose proper transforms are blown up. In other words, ordering the centers blown-up by J_i, the index set of the linear subspace whose proper transform is blown-up, $l_{J_i} \subsetneq l_{J_j}$ only if $i < j$.

This construction gives an explicit basis for $\mathrm{Pic}(\overline{L}_{n-2})$.

Definition 3.3.1. Let $t^{lm} : \overline{L}_{n-2} \to \mathbb{P}^{n-3}$ be the composition of blow-ups in the preceding paragraph. The *Kapranov basis* of $\overline{L}_{n-2}$ consists of the classes of the following divisors in $\overline{L}_{n-2}$:

- the pull-back of a generic hyperplane in $\mathbb{P}^{n-3}$, denoted H^{lm}, and
- the (proper transforms of) exceptional divisors obtained by blowing up (the proper transforms of) l_J for $J \subseteq \{1, \ldots, n-2\}$ and $1 \leq |J| \leq n-4$. We denote these divisors by E_J^{lm}.

The superscript 'lm' is to distinguish these classes from their analogues for $\overline{M}_{0,n}$ to be discussed shortly. Kapranov's basis for $\overline{L}_{n-2}$ implies, in particular, that the Picard number of $\overline{L}_{n-2}$ is $2^{n-2} - n + 1$.

The fan of $\overline{L}_{n-2}$ is determined by the various star subdivisions of the fan for $\mathbb{P}^{n-3}$ as described in Section 3.2 and Example 3.2.6. For $1 \leq |J| \leq n-4$, the ray v_J determines (the proper transform of) the exceptional divisor arising from blowing up (the proper transform of) the coordinate subspace l_J. For $|J| = n-3$, the divisor associated to v_J is the proper transform of the hyperlane l_J.

Example 3.3.2. The Losev-Manin space $\overline{L}_3$ is the blow-up of $\mathbb{P}^2$ in points $l_1 = [1,0,0], l_2 = [0,1,0]$, and $l_3 = [0,0,1]$. The rays of the fan $\Sigma(\overline{L}_3)$ are generated by

$$v_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_{13} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_{23} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix};$$

see Figure 3.2 (b). The fan structure is obvious, but it will be important for the next example to note that a set of rays generates a cone of $\Sigma(\overline{L}_3)$ if and only if the indices of the rays are well-ordered under inclusion. For example, the rays $\{v_1, v_{12}\}$ generate a two-dimensional cone, but $\{v_1, v_2\}$ does not determine a cone.

By the discussion about proper transforms under toric blow-ups above,

$$\begin{aligned}
[D_{23}] &= [H^{lm}] - [E_2^{lm}] - [E_3^{lm}], \\
[D_{13}] &= [H^{lm}] - [E_1^{lm}] - [E_3^{lm}], \\
[D_{12}] &= [H^{lm}] - [E_1^{lm}] - [E_2^{lm}], \\
D_1 = E_1^{lm},\ D_2 &= E_2^{lm}, \text{ and } D_3 = E_3^{lm},
\end{aligned}$$

where we denote by E_i the divisor associated to the ray generated by v_i.

Example 3.3.3. The threefold $\overline{L}_4$ is the blow-up of (the proper transforms of) $l_1 = [1,0,0,0]$, $l_2 = [0,1,0,0]$, $l_3 = [0,0,1,0]$, and $l_4 = [0,0,0,1]$, followed by the blow-ups of the proper transforms of the lines l_{ij}, $1 \le i < j \le 4$. The rays of the fan of $\overline{L}_4$ are generated by the vectors

$$v_{123} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, v_{124} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_{134} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_{234} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix},$$

$$v_{12} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_{13} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_{14} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

$$v_{23} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, v_{24} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, v_{34} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix},$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

See Figure 3.4, where the dashed lines are included only to indicate depth. For the fan structure, a set of rays determines a cone of the fan $\Sigma(\overline{L}_4)$ if and only if the indices of

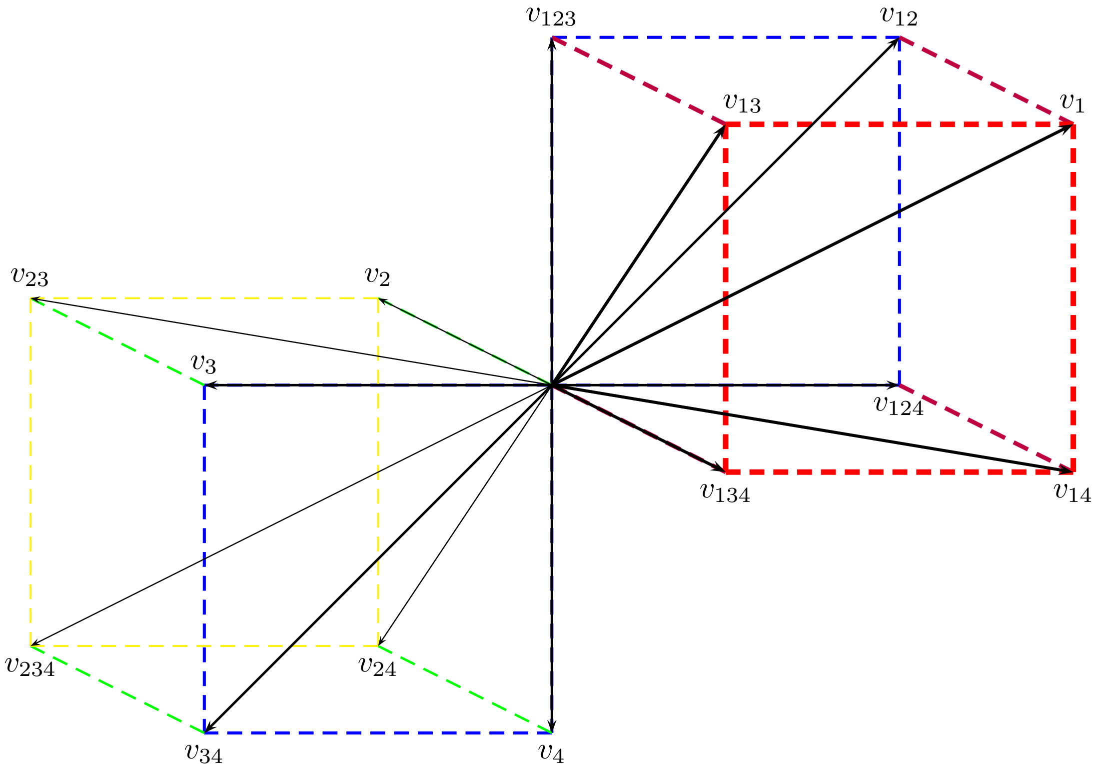


Figure 3.4: Rays of $\overline{L}_4$

the rays are well-ordered under strict inclusion.

As above, we have the following identifications:

$$\begin{aligned}
[D_{234}] &= [H^{lm}] - [E_2^{lm}] - [E_3^{lm}] - [E_4^{lm}] - [E_{23}^{lm}] - [E_{24}^{lm}] - [E_{34}^{lm}],\\
[D_{134}] &= [H^{lm}] - [E_1^{lm}] - [E_3^{lm}] - [E_4^{lm}] - [E_{13}^{lm}] - [E_{14}^{lm}] - [E_{34}^{lm}],\\
[D_{124}] &= [H^{lm}] - [E_1^{lm}] - [E_2^{lm}] - [E_4^{lm}] - [E_{12}^{lm}] - [E_{14}^{lm}] - [E_{24}^{lm}],\\
[D_{123}] &= [H^{lm}] - [E_1^{lm}] - [E_2^{lm}] - [E_3^{lm}] - [E_{12}^{lm}] - [E_{13}^{lm}] - [E_{23}^{lm}],\\
D_1 &= E_1^{lm}, \quad D_2 = E_2^{lm}, \quad D_3 = E_3^{lm}, \quad D_4 = E_4^{lm},\\
D_{12} &= E_{12}^{lm}, \quad D_{13} = E_{13}^{lm}, \quad D_{14} = E_{14}^{lm},\\
D_{23} &= E_{23}^{lm}, \quad D_{24} = E_{24}^{lm}, \quad D_{34} = E_{34}^{lm}.
\end{aligned}$$

There is a slightly different presentation of the blow-up construction of $\overline{L}_{n-2}$ given in [48] and [3], owing to their choice of fan for $\mathbb{P}^d$ as the quotient of the basis vectors $e_1, \ldots, e_{d+1}$ of $\mathbb{C}^{d+1}$ by $(1, \ldots, 1)$.

Kapranov constructed $\overline{M}_{0,n}$ from $\overline{L}_{n-2}$ in [36] by further blow-ups along non-torus-invariant linear subspaces. In addition to linear spans of the points $l_1, \ldots, l_{n-2}$, we take one more point in general position. For concreteness, we set $l_{n-1} = [1, \ldots, 1]$.

We first blow up the proper transform of the point $[1,\ldots,1]$, that is $l_{n-1} \in X_{3\ldots n-2}$, and then blow up the proper transforms of all remaining linear centers containing l_{n-1} in two stages: in the first round of blow-ups, which we call stage-l^c_{n-2}, we blow up proper transforms of linear centers containing l_{n-1} but not l_{n-2} in order of increasing dimension, as above, while in the second, labeled stage-l_{n-2}, we blow-up the remaining proper transforms of linear centers containing both l_{n-2} and l_{n-1}, again, in order of increasing dimension.

Note that this ordering of the blow-ups still respects the partial ordering by inclusion.

Definition 3.3.4. Let $f : \overline{M}_{0,n} \to \overline{L}_{n-2}$ be the composition of blow-ups involving l_{n-1} above, and set $t = t^{lm} \circ f : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$.

The *Kapranov basis* of $\overline{M}_{0,n}$ consists of the classes of the following divisors in $\overline{M}_{0,n}$:

- the pullback under t of a generic hyperplane in $\mathbb{P}^{n-3}$, denoted by H;
- the proper transforms of E^{lm}_J, where $J \subseteq \{1,\ldots,n-2\}$, $1 \leq |J| \leq n-4$; and
- the (proper transforms of the) exceptional divisors obtained by blowing up the proper transforms of l_J, where $n-1 \in J$ and $1 \leq |J| \leq n-4$.

We denote divisors of the last two types by E_J.

Despite the different order of blow-ups above compared to Section 1.3.3, we will show in Proposition 3.3.12 that, up to isomorphism, there is only one 'Kapranov' basis.

We next establish how the members of the Kapranov basis for $\overline{M}_{0,n}$ relate to the pull-backs of classes in the Kapranov basis for $\overline{L}_{n-2}$ under the composition of blow-ups $f : \overline{M}_{0,n} \to \overline{L}_{n-2}$. Before turning to the general case, we look at the simplest non-trivial example of $\overline{M}_{0,5}$ and $\overline{L}_3$.

Example 3.3.5. To obtain $\overline{M}_{0,5}$ from $\overline{L}_3$, we further blow-up $p_4 = [1,1,1]$. Since E_4 is disjoint from $f^*(E^{lm}_i)$, $i = 1,\ldots,3$, it follows that the proper transforms of the E^{lm}_i equal their pull-backs under $f : \overline{M}_{0,5} \to \overline{L}_3$ for $i = 1,2,3$, that is,

$$E_i = f^*(E^{lm}_i), \quad i = 1,\ldots,3.$$

For $n \geq 6$, the exceptional divisors from stages l^c_{n-2} and l_{n-2} are not disjoint from pull-backs of exceptional divisors of the preceding stages. Nevertheless, we show that the same relationship between pull-backs and proper transforms of exceptional divisors holds.

Proposition 3.3.6. *For every $J \subseteq \{1,\ldots,n-2\}$, $2 \leq |J| \leq n-4$,*

$$f^*(E^{lm}_J) = \widetilde{E^{lm}_J} = E_J.$$

Note that the second equality is definitional. For the proof we use a general characterization of proper transforms from [23], B.6:

Proposition 3.3.7. *Let Z be a smooth subvariety of a variety Y, and let $f_Z : Bl_Z(Y) \to Y$ be the blow-up of Y along Z. If V is a smooth subvariety of Y containing Z, then the proper transform $\tilde{V}$ is the blow-up of V along Z, that is, $\tilde{V} = Bl_{V\cap Z}V \to V$.*

A second ingredient is the notion of *clean* intersections, as formulated in [47]. Let X be a nonsingular variety, and let A and B be nonsingular subvarieties. Denote by T_A the total space of the tangent bundle of A, here considered as a subbundle of T_X. For $a \in A$ we denote by $T_{A,a}$ tangent space of A at the point a, taken as a subspace of $T_{X,a}$.

Definition 3.3.8. The subvarieties A and B are said to intersect *cleanly* if

(i) the set-theoretic intersection $A \cap B$ is a nonsingular subvariety of X, and

(ii) $T_{A\cap B,y} = T_{A,y} \cap T_{B,y}$ for all $y \in A \cap B$.

For example, two lines l and l' in $\mathbb{P}^3$ will always intersect cleanly, even though their intersection is never transverse: if l and l' are skew, then they satisfy the definition of clean intersection trivially, while if l and l' meet at a point x, the intersection $T_{l,x} \cap T_{l',x}$ is the trivial vector space, which is the tangent space to the subvariety x, and finally, if $l = l'$, the criteria for clean intersection are clearly satisfied. More generally, if l_J and $l_{J'}$ are linear subspaces of $\mathbb{P}^m$, their intersection is also clean.

Clean intersections behave nicely under blow-ups. The following is from Lemma 2.9 in [47]. Note that will now denote the proper transforms by a $\sim$; this is one of the exceptions to suppressing the proper transform mentioned in Notation 3.2.7.

Lemma 3.3.9. *For A, B, C, and F nonsingular subvarieties of X, a nonsingular variety, let $f_F : Bl_F(X) \to X$ be the blow-up of X along F with exceptional divisor E.*

(i) If A and B intersect cleanly, with $A \not\subseteq B$, $B \not\subseteq A$ such that $F = A \cap B$, then $\widetilde{A} \cap \widetilde{B} = \emptyset$.

(ii) If $A \supsetneq F$, then $\widetilde{A}$ and E intersect transversally.

(iii) If $F \subseteq A$ with both A and F intersecting B transversally, then $\widetilde{A}$ and $\widetilde{B}$ intersect transversally.

Proof of Propostion 3.3.6. Since the pull-back and proper transform of a subvariety coincide if the blow-up is along a center disjoint from the subvariety, we may restrict attention to subspaces l_J and $l_{J'}$ such that $J \cap J' \neq \emptyset$. By Lemma 3.3.9 (i), if $J \not\subseteq J'$ and $J' \not\subseteq J$, then once the proper transform of $l_{J\cap J'}$ is blown-up, the proper transforms of l_J and $l_{J'}$ will be disjoint, as will all successive proper transforms and inverse images of l_J and $l_{J'}$. Due to the partial ordering of blow-ups, we therefore need only consider how the pull-back and proper transform of the exceptional divisor of the blow-up of (the proper transform of) l_J relate under the blow-up of the proper transform of $l_{J'}$ for $J \subsetneq J'$.

We first consider $J_1 \subseteq J_2 \subseteq J_3$, with $|J_3| = |J_2|+1 = |J_1|+2$. Let $f_{J_2} : X_{J_2} \to X_{J_1}$ be the blow-up along $l_{J_2} \subseteq X_{J_1}$ with exceptional divisor E. For the blow-up $f_{J_3} : X_{J_3} \to$

X_{J_2} along $l_{J_3} \subseteq X_{J_2}$, we want to show that $f_{J_3}^*(E) = \widetilde{E}$. The generic point of E is disjoint from the center of the blow-up l_{J_3}, so since f_{J_3} is an isomorphism over such points, it suffices to show that $f_{J_3}^{-1}(E) = \widetilde{E}$.

By Proposition 3.3.7, $\widetilde{E}$ is the blow-up of E along $l_{J_3} \cap E$. Denote this blow-up by $\phi : \widetilde{E} \to E$, and its exceptional divisor by F. Since $\phi^{-1}(E \setminus l_{J_3}) = f_{J_3}^{-1}(E \setminus l_{J_3}) = f_{J_3}^{-1}(E) \setminus f_{J_3}^{-1}(l_{J_3})$, we are finished in this case if $F = f_{J_3}^{-1}(l_{J_3} \cap E)$. But $F \subseteq f_{J_3}^{-1}(l_{J_3} \cap E)$, and F is a projective bundle over $l_{J_3} \cap E$, with each fiber a projective space of dimension equal to the codimension of $l_{J_3} \cap E$ in E. By Lemma 3.3.9 (ii), E meets l_{J_3} transversely, so for $q \in l_{J_3} \cap E$, the fiber F_q has dimension $n - |J_3| - 4$. On the other hand, $f_{J_3}^{-1}(q)$ is a fiber of the projectivized normal bundle $\mathbb{P}(\mathcal{N}_{X_{J_2} \setminus l_{J_3}})$. The dimension of $f_{J_3}^{-1}(q)$ is also $n - |J_3| - 4$, so we have an inclusion of projective spaces of the same dimension, giving $F_q = f_{J_3}^{-1}(q)$. It follows that $F = f_{J_3}^{-1}(l_{J_3} \cap E)$, as desired.

We now consider a chain of inclusions $J_1 \subseteq J_2 \subseteq J_3 \subseteq \ldots \subseteq J_k$, where again, $|J_j| = |J_{j-1}| + 1$. Abusing notation as usual, let $E \subseteq X_{J_{k-1}}$ be the proper transform of the exceptional divisor of $f_{J_2} : X_{J_2} \to X_{J_1}$. If $\widetilde{E}$ is now the proper transform under $f_{X_{J_k}} : X_{J_k} \to X_{J_{k-1}}$, the blow up along $l_{J_k} \subseteq X_{J_{k-1}}$, the proposition is proved once we show $\widetilde{E} = f_{J_k}^{-1}(E)$. Since every center blown up before l_{J_k} is contained in l_{J_k}, Lemma 3.3.9 (iii) implies that $\widetilde{E}$ and l_{J_k} intersect transversally. The rest of the proof now proceeds identically to the initial case. □

Since the hyperplane class in $\overline{M}_{0,n}$ is by definition the pull-back of the hyperplane class on $\overline{L}_{n-2}$, Proposition 3.3.6 implies:

Corollary 3.3.10. *The pull-back of* $[D] = h[H^{lm}] + \sum_{\substack{1 \leq |J| \leq n-4, \\ J \subseteq \{1,\ldots,n-2\}}} e_J[E_J^{lm}] \in \operatorname{Pic}(\overline{L}_{n-2})$ *to* $\operatorname{Pic}(\overline{M}_{0,n})$ *is given by*

$$f^*[D] = h[H] + \sum_{\substack{1 \leq |J| \leq n-4 \\ J \subseteq \{1,\ldots,n-2\}}} e_J[E_J].$$

Proposition 3.3.6 almost gives an identification between the divisors D_J of $\overline{L}_{n-2}$ and the boundary divisors Δ_J of $\overline{M}_{0,n}$. The minor obstacle to this identification is that the order of blow-ups used in the majority of the literature is not the one we used in defining the Kapranov basis (Definition 3.3.4); instead the ordering due to Hassett is generally used (see Section 1.3.3 and [32]). Hence it is not immediately obvious (but also not difficult to prove) that the hyperplane classes and exceptional divisors arising from the different blow-up orderings are interchangeable.

For the remainder of this section only, we distinguish the varieties resulting from the two orderings of the blow-ups by $\overline{M}_{0,n}^k$ and $\overline{M}_{0,n}^h$ for the Kapranov and Hassett constructions, respectively. Likewise, we denote the resulting bases of the Picard groups by $\mathcal{B}^k = \{[H^k], [E_J^k]\}$ and $\mathcal{B}^h = \{[H^h], [E_J^h]\}$. In Chapter 1, we described a dictionary between boundary divisors and exceptional divisors from the blow-up construction without specifying which blow-up construction was intended. We now prove that the two ordering of the blow-ups are related by an isomorphism that takes the basis $\mathcal{B}^h$ to the basis $\mathcal{B}^k$.

We begin with a basic observation.

Lemma 3.3.11. *Let X be a smooth variety with disjoint, closed subvarieties A_1 and A_2. Let $f_1 : X_1 \to X$ be the blow-up of X along A_1, and let $f_2 : X_2 \to X$ be the blow-up of X along A_2. We denote the proper transform of A_2 under f_1 by $\widetilde{A_2}^{f_1}$, and likewise, the proper transform of A_1 under f_2 by $\widetilde{A_1}^{f_2}$. Let $g_2 : X_{21} \to X_1$ be the blow-up of X_1 along $\widetilde{A_2}^{f_1}$, and let $g_1 : X_{12} \to X_2$ be the blow-up of X_2 along $\widetilde{A_1}^{f_2}$.*

If E_2 is the exceptional divisor of g_2, and E_1 the proper transform under g_2 of the exceptional divisor of f_1, and likewise F_1 is the exceptional divisor of g_1 and F_2 the proper transform under g_1 of the exceptional divisor of f_2, then there exists an isomorphism

$$\phi : X_{21} \overset{\cong}{\to} X_{12}.$$

such that

$$\phi^*(F_i) = E_i \text{ for } i = 1, 2.$$

The first part of the lemma is a standard result, and is proved in greater generality in [47]. It is the second part that enables us to prove that the Kapranov and Hassett orderings result in the same basis for $\mathrm{Pic}(\overline{M}_{0,n})$ up to isomorphism.

Proof. First, since A_1 and A_2 are disjoint, $\widetilde{A_j}^{f_i} = f_i^{-1}(A_j)$ for $i, j = 1, 2$, $i \neq j$. The proof involves finding open covers of X_{12} and X_{21}, plus isomorphisms of the elements of the covers that agree on overlap.

$$\begin{tikzcd}
X_{21} = Bl_{\widetilde{A_2}^{f_1}}(X_1) \arrow[rr, dotted, "\phi_i"] \arrow[d, "g_2"] & & X_{12} = Bl_{\widetilde{A_1}^{f_2}}(X_2) \arrow[d, "g_1"] \\
X_1 = Bl_{A_1}(X) \arrow[dr, "f_1"'] & & X_2 = Bl_{A_2}(X) \arrow[dl, "f_2"] \\
& X &
\end{tikzcd}$$

Define $U_1 = X_{21} \setminus (g_2^{-1} \circ f_1^{-1}(A_1))$ and $U_2 = X_{21} \setminus (g_2^{-1} \circ f_1^{-1}(A_1))$. Likewise, set $V_1 = X_{12} \setminus (g_1^{-1} \circ f_2^{-1}(A_1))$ and $V_2 = X_{12} \setminus (g_1^{-1} \circ f_2^{-1}(A_2))$. Then $\{U_1, U_2\}$ and $\{V_1, V_2\}$ define open covers X_{21} and X_{12}, respectively.

Note that $g_2|_{U_1} : U_1 \to X_1 \setminus f^{-1}(A_1)$ is the blow-up of $X_1 \setminus f_1^{-1}(A_1)$ along $f_1^{-1}(A_2)$, and f_1 defines an isomorphism between $X_1 \setminus f_1^{-1}(A_1)$ and $X \setminus A_1$. Since $g_2^{-1} \circ f_1^{-1}(A_2) = E_2$ is a Cartier divisor, by the proof of the universal property of blowing-up (see [31]), the unique morphism between U_1 and $X_2 \setminus f_2^{-1}(A_1)$ factoring f_2 is an isomorphism. Composing with the inverse of the isomorphism $g_1|_{V_1} : V_1 \to X_2 \setminus (f_2^{-1}(A_1))$ determines an isomorphism $\phi_1 : U_1 \to V_1$. We similarly obtain an isomorphism $\phi_2 : U_2 \to V_2$.

By definition, $(f_1 \circ g_2)|_{U_1 \cap U_2} : U_1 \cap U_2 \to X \setminus (A_1 \cup A_2)$ is an isomorphism, as is $(f_2 \circ g_1)|_{V_1 \cap V_2} : V_1 \cap V_2 \to X \setminus (A_1 \cup A_2)$. Moreover, $(f_2 \circ g_1)^{-1} \circ (f_1 \circ g_2)|_{U_1 \cap U_2}$ agrees with ϕ_1 and ϕ_2 on $U_1 \cap U_2$. Hence ϕ_1 and ϕ_2 glue together to give the desired isomorphism ϕ. By construction, $\phi^{-1}(F_i) = E_i$ for $i = 1, 2$. $\square$

Recall that Hassett's ordering of blow-ups follows the dimension of the linear centers in $\mathbb{P}^{n-3}$. Specifically, we first blow-up the points (or proper transforms of) $l_1, \dots, l_{n-1}$, then the proper transforms of the lines $l_{12}, \dots, l_{n-2\,n-1}$, continuing until all proper transforms of l_J with $|J| = n - 4$ have been blown up.

Proposition 3.3.12. *There exists an isomorphism* $\phi : \overline{M}^k_{0,n} \to \overline{M}^h_{0,n}$ *such that*

$$\phi^*(H^h) = H^k \text{ and } \phi^*(E^h_J) = E^k_J.$$

The existence of such an isomorphism is well-known (see [32] and [47]), and the claim about pull-backs of basis elements could be proven by a modular interpretation of the Kapranov basis, but a direct proof seems preferable.

Proof of Proposition 3.3.12. In the Kapranov ordering, l_{n-1} is disjoint from all linear subspaces l_J whose proper transforms are blown-up before it. Hence all proper transforms of l_{n-1} and such l_J are also disjoint, so by Lemma 3.3.11, we may interchange the blowing-up of the proper transform of l_{n-1} successively with each of the blow-ups preceding it. In particular, we may blow up l_{n-1} after blowing up l_{n-2} to match the Hassett ordering.

In general, suppose inductively that we have brought the proper transform of the i^{th} linear subspace l_{J_i} into agreement with the Hassett ordering for $i < j$. For l_{J_i}, $i < j$, such that $|J_i| \geq |J_j|$, we have $l_{J_i} \nsubseteq l_{J_j}$ and $l_{J_j} \nsubseteq l_{J_i}$. Lemma 3.3.9 implies that, after blowing up the proper transform of $l_{J_i \cap J_j}$, the proper transforms of l_{J_i} and l_{J_j} are disjoint. Since $|J_i \cap J_j| < |J_j|$, we may switch the order of blow-ups of the proper transform of l_{J_j} successively with each l_{J_i} such that $|J_i| \geq |J_j|$. In particular, we may change the order so that the proper transform of l_{J_j} is as in the Hassett ordering. Applying Lemma 3.3.11 proves the claim about the respective exceptional divisors. The claim about pull-backs of a generic hyperplane in $\mathbb{P}^{n-3}$ follows from the result about exceptional divisors plus the gluing of Lemma 3.3.11. $\square$

Via Propositions 3.3.6 and 3.3.12, we can identify the divisors D_J in $\overline{L}_{n-2}$ corresponding to rays generated by the v_J with boundary divisors Δ_J in $\overline{M}_{0,n}$.

Definition 3.3.13. For $J \subseteq \{1, \dots, n-2\}$, $1 \leq |J| \leq n-3$, let $\Delta^{lm}_{J \cup \{n\}}$ be the torus-invariant divisor $\Delta^{lm}_{J \cup \{n\}} = V(\langle v_J \rangle_{\geq 0})$.

If $1 \leq |J| \leq n-4$, then $\Delta^{lm}_{J \cup \{n\}} = E_J$, while for $|J| = n-3$, $\Delta^{lm}_{J \cup \{n\}}$ is the proper transform of the line containing all $l_{J'} \subseteq \mathbb{P}^{n-3}$, with $J' \subsetneq J$. As with boundary divisors in $\overline{M}_{0,n}$, we will identify Δ^{lm}_J and $\Delta^{lm}_{J^c}$.

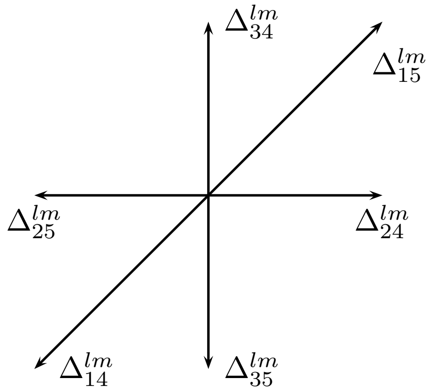


Figure 3.5: Rays of $\overline{L}_3$ labeled by boundary divisors

Corollary 3.3.14. *For every boundary divisor class $[\Delta_J^{lm}] \in \mathrm{Pic}(\overline{L}_{n-2})_{\mathbb{Q}}$,*

$$f^*([\Delta_J^{lm}]) = [\Delta_J] \in \mathrm{Pic}(\overline{M}_{0,n})_{\mathbb{Q}}.$$

This identification for $n = 5$ and 6 (see Examples 3.3.2 and 3.3.3) is depicted in Figures 3.5 and 3.6. Note that, as in Figure 3.4 of Example 3.3.3, the dashed lines in Figure 3.6 are there to indicate depth.

The final ingredient required to prove Theorem 3.1.1 is an isomorphism between the global sections of a divisor in $\overline{L}_{n-2}$ and the global sections of its pull-back in $\overline{M}_{0,n}$. That the induced map on global sections is an isomorphism is a standard result, and holds also for much more general situations (see Theorem 2.31 of [35]).

Lemma 3.3.15. *For $f : \overline{M}_{0,n} \to \overline{L}_{n-2}$ as above, and D a divisor on $\overline{L}_{n-2}$,*

$$f^* : H^0(\overline{L}_{n-2}, D) \overset{\cong}{\to} H^0(\overline{M}_{0,n-2}, f^*(D)).$$

Proof. We first set $L = \mathcal{O}_{\overline{L}_{n-2}}(D)$ for the locally-free sheaf (i.e. line bundle) determined by the divisor D ($\overline{L}_{n-2}$ is smooth, so Weil and Cartier divisors coincide). By the projection formula, $f_*f^*L \cong (f_*\mathcal{O}_{\overline{M}_{0,n}}) \otimes L$, but $f_*(\mathcal{O}_{\overline{M}_{0,n}}) = \mathcal{O}_{\overline{L}_{n-2}}$, as in the proof of Zariski's main theorem (see [31], Corollary III.11.4). It follows that $H^0(\overline{M}_{0,n}, f^*(L)) \cong H^0(\overline{L}_{n-2}, f_*f^*(L)) \cong H^0(\overline{L}_{n-2}, L)$. □

Thus far, we have been discussing a single Losev-Manin moduli space $\overline{L}_{n-2}$, where, in relation to $\overline{M}_{0,n}$, we take the points labeled by $(n-1)$ and n to be the poles (or, to link the modular and blow-up descriptions, the $(n-1)^{\mathrm{st}}$ marked point corresponds to the non-toric point $l_{n-1} \in \mathbb{P}^{n-3}$, and the n^{th} marked point is the 'moving' point). The choice of poles, however, is arbitrary, since permuting the marked points results in isomorphic copies of $\overline{M}_{0,n}$. Hence there are $\binom{n}{2}$ Losev-Manin moduli spaces that have the same relationship with $\overline{M}_{0,n}$ as that described in this section by choosing different pairs of points for the poles.

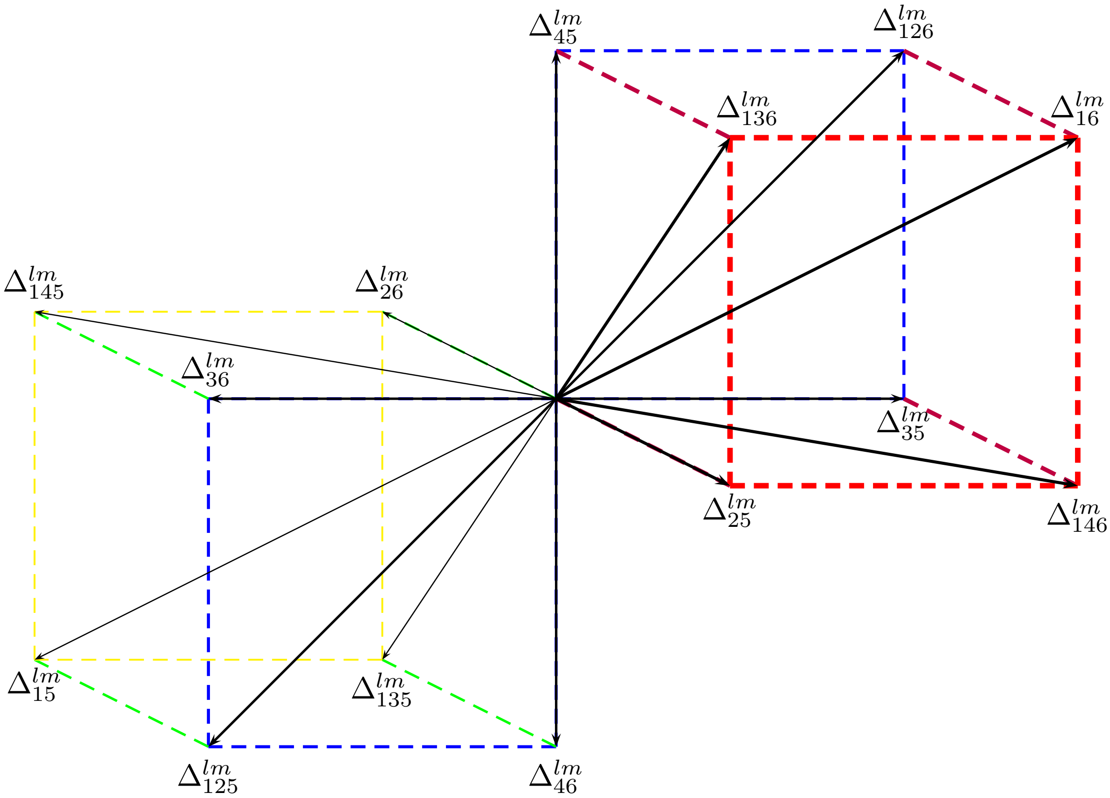


Figure 3.6: Rays of $\overline{L}_4$ labeled by boundary divisors

We denote by $\overline{L}_{n-2}(i,j)$ the Losev-Manin space with poles the i^{th} and j^{th} marked points. Let $f(i,j) : \overline{M}_{0,n} \to \overline{L}_{n-2}(i,j)$ be composition of blow-ups described in Section 3.3. To select divisors whose classes give a basis of $\operatorname{Pic}(\overline{L}_{n-2}(i,j))$, we take all divisors Δ_J^{lm} such that $J \subseteq (\{1,\dots,n\} \setminus \{i\})$, with $j \in J$, and $2 \leq |J| \leq n-3$, plus the pull-back under $f(i,j)$ of a generic hyperplane in $\mathbb{P}^{n-3}$, denoted by H. In the following proposition, we set $r = 2^{n-2} - n + 1$, and we identify a lattice point $\mathbf{m}(i,j) \in \mathbb{Z}^r$ with a divisor class in $\operatorname{Pic}(\overline{L}_{n-2}(i,j))$ by setting $\mathbf{m}(i,j) = (m_h, m_J : J \subseteq \{1,\dots,n\} \setminus \{j\},\ i \in J,\ 2 \leq |J| \leq n-3)$.

Proposition 3.3.16. *The induced map $f(i,j)^* : \operatorname{Cox}(\overline{L}_{n-2}(i,j)) \to \operatorname{Cox}(\overline{M}_{0,n})$ induces an isomorphism of graded $\mathbb{C}$-algebras,*

$$\operatorname{Cox}(\overline{L}_{n-2}(i,j)) \xrightarrow{\cong} \bigoplus_{\boldsymbol{m}(i,j) \in \mathbb{Z}^r} H^0(\overline{M}_{0,n}, \mathcal{O}(m_h H + \sum_J m_J \Delta_J)),$$

defined by $x_{\Delta_J^{lm}} \mapsto x_{\Delta_J}$.

Proof. The proposition follows from Corollary 3.3.10, Proposition 3.3.12, and Proposition 3.3.6. □

Since $\operatorname{Cox}(\overline{L}_{n-2}(i,j))$ is a polynomial ring in the variables $x_{\Delta_J^{lm}}$, where $i \in J$, $J \subseteq (\{1,\dots,n\} \setminus \{j\})$, and $2 \leq |J| \leq n-3$, we obtain as a corollary Theorem 3.1.1.

As a direct application, we can reduce computing h^0 of all divisor classes that appear on the right hand side of the isomorphism in Proposition 3.3.16 to counting integral points of the corresponding polytope, as defined in Equation 3.2.1.

3.4 Relations among generators in the Cox ring of $\overline{M}_{0,n}$

To find collections of relations among boundary sections in $\operatorname{Cox}(\overline{M}_{0,n})$, we consider divisor classes whose corresponding complete linear systems determine morphisms to a projective space that are a composition of a forgetful morphism $\pi_J : \overline{M}_{0,n} \to \overline{M}_{0,\{1,\dots,n\}\setminus J}$ with one of the Kapranov morphisms $t_m : \overline{M}_{0,\{1,\dots,n\}\setminus J} \to \mathbb{P}^{n-|J|-3}$, where $m \notin J$ (in contrast to Definition 3.3.1, we now keep track of which marked point is the moving point; see also Section 1.3.3). For such a divisor $F_{J,m}$, we will show that the $[F_{J,m}]$-graded part of $\operatorname{Cox}(\overline{M}_{0,n})$ can be generated by $\binom{n-|J|-1}{2}$ boundary sections, but $h^0(\overline{M}_{0,n}, F_{J,m}) = n - |J| - 2$, thus giving non-trivial relations for every $n \geq 5$.

First, we fix some notation and make a few further remarks about the Kapranov construction. Since we will be 'forgetting' varied subsets $J \subseteq \{1,\dots,n\}$, we will keep track of which points are remembered by labeling the target space as $\overline{M}_{0,\{1,\dots,n\}\setminus J}$. Secondly, Kapranov showed in [37] that the morphism $t_m : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ is induced by the psi-class ψ_m of the moving point (see Section 1.3.3). In particular, the hyperplane class of the Kapranov basis equals ψ_m. If we instead choose the k^{th} marked point to be the moving one, the composition of blow-ups is now induced by ψ_k. Different choices of moving points relate to one another via a Cremona transformation of $\mathbb{P}^{n-3}$, that is, for $i \neq j$, $\psi_i \circ \psi_j : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-3}$ is a Cremona transformation (see [37]).

To define the divisor classes $F_{J,m}$, fix a subset $J \subseteq \{1, \ldots, n\}$ with $0 \leq |J| \leq n-4$, and choose $m \notin J$. For simplicity, we will choose $J \subseteq \{1, \ldots, n-1\}$, and take $k = n$, since the general situation can be obtained from this one by a Cremona transformation. In particular, for $i \neq j$, the change of basis on $\mathrm{Pic}(\overline{M}_{0,n})$ is given by

$$\psi_j = (n-3)\psi_i - \sum_{\substack{J \subseteq (\{1,\ldots,n\} \setminus \{i,j\}) \\ 1 \leq |J| \leq n-4}} (n - |J| - 3)[\Delta_{J \cup \{i\}}], \tag{3.4.1}$$

$$[\Delta_{J\cup\{j\}}] = \begin{cases} [\Delta_{J\cup\{j\}}] & \text{if } i \in J, \\ [\Delta_{(J\cup\{j\})^c}] & \text{if } i \notin J, |J| > 1, \\ \psi_i - \sum\limits_{T \subsetneq (J\cup\{i,j\})^c} [\Delta_{T\cup\{i\}}] & \text{if } i \notin J \text{ and } |J| = 1 \end{cases}$$

(see [9] for the first formula, though it also follow easily from the expressions for psi-classes determined in [2]).

Define $\phi_{J,n} = t_n \circ \pi_J : \overline{M}_{0,n} \to \mathbb{P}^{n-|J|-3}$; this morphism is induced by the divisor class $[F_{J,n}] = \phi^*_{J,n}(\mathcal{O}_{\mathbb{P}^{n-|J|-3}}(1))$. We can thus determine divisor representatives by studying how the hyperplane class on $\mathbb{P}^{n-|J|-3}$ pulls-back under $\phi_{J,n}$, or, equivalently, how psi-classes pull-back under the forgetful morphisms π_J. We begin with a lemma from [2].

Lemma 3.4.1. *For $q \in P \subseteq \{1, \ldots, n\}$ and $p \in P - \{q\}$ the pull-back of ψ_p under $\pi_q : \overline{M}_{0,P} \to \overline{M}_{0,P\setminus\{q\}}$ is given by*

$$\pi^*_q(\psi_p) = \psi_p - [\Delta_{pq}] \tag{3.4.2}$$

A second fact of use is a special case of a lemma of Keel from [38].

Lemma 3.4.2. *For $q \in P \subseteq \{1, \ldots, n\}$ and $T \subseteq P \setminus \{q\}$, the pullback of $[\Delta_T]$ under $\pi_q : \overline{M}_{0,P} \to \overline{M}_{0,P\setminus\{q\}}$ is*

$$\pi^*_q([\Delta_T]) = [\Delta_T] + [\Delta_{T\cup\{q\}}]. \tag{3.4.3}$$

The final ingredient is that if $J = \{j_1, \ldots, j_k\}$ (allowing the possibility $J = \emptyset$), then the forgetful morphism π_J decomposes as $\pi_J = \pi_{j_1} \circ \ldots \circ \pi_{j_k}$ (or as any re-ordering). Combining, we obtain the following formula.

Lemma 3.4.3. *For every $J \subseteq \{1, \ldots, n-1\}$, the pullback of ψ_n under $\pi_J : \overline{M}_{0,\{1,\ldots,n\}} \to \overline{M}_{0,\{1,\ldots,n\}\setminus J}$ is*

$$\pi^*_J(\psi_n) = \psi_n - \sum_{T \subseteq J} [\Delta_{T\cup\{n\}}] \tag{3.4.4}$$

Proof. The case $|J| = 1$ is Lemma 3.4.1 of Arbarello and Cornalba, so we assume inductively the validity of Equation (3.4.4) for all $J \subseteq \{1, \ldots, n-1\}$, $|J| = k < n-1$. Let

$q \in \{1, \ldots, n-1\} \setminus J$. Then

$$\begin{aligned}
\pi^*_{J\cup\{q\}}(\psi_n) &= \pi^*_q \circ \pi^*_J(\psi_n) \\
&= \pi^*_q(\psi_n - \sum_{T\subseteq J}[\Delta_{T\cup\{n\}}]) \\
&= \psi_n - [\Delta_{qn}] - \sum_{T\subseteq J}([\Delta_{T\cup\{n\}}] + [\Delta_{T\cup\{q,n\}}]) \\
&= \psi_n - \sum_{T\subseteq (J\cup\{q\})}[\Delta_{T\cup\{n\}}],
\end{aligned}$$

as desired. □

Therefore, with respect to the Kapranov basis,

$$[F_{J,n}] = [H] - \sum_{T\subseteq J}[E_T]. \tag{3.4.5}$$

Before considering how to represent the divisor classes $[F_{J,n}]$ as effective sums of boundary, we show that these classes give a basis for $\mathrm{Pic}(\overline{M}_{0,n})_{\mathbb{Q}}$.

Proposition 3.4.4. *The collection of divisor classes,*

$$\{[F_{J,n}] : J \subseteq \{1, \ldots, n-1\}, 0 \leq |J| \leq n-4\},$$

determines a basis of $\mathrm{Pic}(\overline{M}_{0,n})_{\mathbb{Q}}$.

Proof. The first thing to note is that the number of divisor classes $[F_{J,n}]$ is

$$\begin{aligned}
1 + \binom{n-1}{2} + \binom{n-1}{3} + \ldots + \binom{n-1}{n-4} &= 2^{n-1} - \binom{n-1}{n-3} - \binom{n-1}{n-2} - \binom{n-1}{n-1} \\
&= 2^{n-1} - \binom{n}{2} - 1,
\end{aligned}$$

which is precisely the dimension of $\mathrm{Pic}(\overline{M}_{0,n})_{\mathbb{Q}}$. It therefore suffices to show that these classes are linearly dependent.

To show linear independence, we set $\rho = \dim \mathrm{Pic}(\overline{M}_{0,n})_{\mathbb{Q}}$ and fix an isomorphism $\mathrm{Pic}(\overline{M}_{0,n})_{\mathbb{Q}} \cong \mathbb{R}^\rho$ by choosing the Kapranov basis, ordered in the usual way:

$$\mathcal{B}_K = \{[H], [E_1], \ldots, [E_{n-1}], [E_{12}], [E_{13}], \ldots [E_{4\ldots n-1}]\}.$$

We similarly order the set of classes $[F_{J,k}]$ as

$$\mathcal{B}_F = \{[H], [H] - [E_1], [H] - [E_2], \ldots, [H] - [E_4] - [E_5] - \ldots - [E_{n-1}] - \ldots - [E_{4\ldots n-1}]\}.$$

Writing each element of $\mathcal{B}_F$ in the coordinates given by the Kapranov basis $\mathcal{B}_K$, we see that the i^{th} coordinate of the i^{th} element of $\mathcal{B}_F$ will be -1 (except for $i = 1$, when the coordinate is just 1), with all j^{th} coordinates, $j > i$, equal to 0. Hence the matrix whose columns are the coordinate vectors of the $[F_{J,n}]$ is upper triangular, with the (i,i) entry equal to 1 if $i = 1$, and -1 otherwise, thus showing that the $[F_{J,n}]$ are linearly independent. $\square$

Finally, to obtain integral effective sums of boundary divisors whose class is $[F_{J,n}]$, we vary the representative of $[H]$, as in the dictionary of Equations 1.3.3. Specifically, for every choice of distinct $a, b \in \{1, \dots, n-1\} \setminus J$, substituting $[H]$ as in the expression for $[\Delta_{ab}]$ of Equation (1.3.4) gives $[F_{J,n}]$ as an effective sum of boundary classes. Moreover, any substitution for one of the $\Delta_{T\cup\{n\}}$ appearing in Equation 3.4.5 via a Keel relation increases the number of boundary divisors with a negative coefficient by one. It follows that there are as many ways to write $[F_{J,n}]$ as an effective sum of boundary divisor as there are ways to express the class of H in the dictionary of Equation (1.3.4) as a sum of boundary involving Δ_{ab} for $\{a,b\} \subseteq \{1, \dots, n-1\} \setminus J$, namely, in $\binom{n-|J|-1}{2}$ ways.

Moreover, since each class $[F_{J,n}]$ determines a morphism to $\mathbb{P}^{n-|J|-3}$, there are $n - |J| - 2$ independent global sections of $[F_{J,n}]$. We have thus proved Theorem 3.1.2. Note that if we restrict to $|J| = n-4$, then the $[F_{J,n}]$ are precisely the Keel classes studied in Chapter 2. Moreover, it is shown in [4] that for $\overline{M}_{0,5}$, all relations are generated by these classes (see also [43]).

4 The complete intersection cone of $\overline{M}_{0,6}$

4.1 Introduction

A foundational result in the geometry of projective varieties is Kleiman's theorem, proved in [40], which states the closure of the ample cone equals the nef cone. One of the inclusions is straightforward to prove. If D is ample, then for a sufficiently large integer m, the complete linear series $|mD|$ induces an embedding into some projective space, $\phi_{|mD|} : X \hookrightarrow \mathbb{P}^N$, such that $\mathcal{O}_X(D)$ is the pull-back of the hyperplane class $\mathcal{O}_{\mathbb{P}^N}(1)$. For an effective one-cycle class $\gamma \in \overline{\mathrm{NE}}(X)$, there exist a finite number of classes γ_i of irreducible curves in X such that $\gamma = \sum a_i\gamma_i$, with $a_i \geq 0$. To see that $D \cdot \gamma \geq 0$, note first that $D \cdot \gamma = \mathcal{O}_X(D) \cdot \gamma = \phi^*_{|mD|}(\mathcal{O}_{\mathbb{P}^N}(1)) \cdot \gamma$, which, via the projection formula, equals $\mathcal{O}_{\mathbb{P}^N}(1) \cdot (\phi_{|mD|})_*(\gamma)$, or, by linearity,

$$\begin{aligned} D \cdot \gamma &= \mathcal{O}_{\mathbb{P}^N}(1) \cdot (\phi_{|mD|})_*(\sum a_i\gamma_i) \\ &= \sum a_i(\mathcal{O}_{\mathbb{P}^N}(1) \cdot (\phi_{|mD|})_*(\gamma_i)). \end{aligned}$$

Since the γ_i are represented by irreducible curves, so are the push-forwards under $\phi_{|mD|}$, and so for each i we can find a hyperplane in $\mathbb{P}^N$ meeting this representative curve transversely, hence $D \cdot \gamma$ is non-negative. Since the nef cone is by definition closed, this shows that the closure of the ample cone is contained in the nef cone. The proof of the opposite inclusion is more involved; see [40], or Section 1.4.C of [45].

Via duality, Kleiman's theorem can be restated as the equality of cones $\overline{\mathrm{NE}}(X)^\vee = \overline{\mathrm{Amp}}(X)$. It is natural to wonder which other cones of divisor and curves fit into a Kleiman-type duality. For the pseudoeffective cone of divisor classes, it is not difficult to see that dual cone $\overline{\mathrm{Eff}}(X)^\vee$ contains the closure of the cone of movable curve classes, where a reduced, irreducible curve C is called a *movable curve* if $C = C_{t_0}$ belongs to an algebraic family $(C_t)_{t\in S}$ covering X. To see that classes of movable curves are contained in $\overline{\mathrm{Eff}}(X)^\vee$, let D be an effective prime divisor, and let C be a movable curve. Since the support of D is a codimension one subvariety, there must exist an irreducible curve C' in the covering family containing C such that C' is not contained in the support of D, hence $C' \cdot D \geq 0$. Since algebraic equivalence is finer than numerical equivalence, it follows that $C \cdot D \geq 0$. The other inclusion was proved in 2004 by Boucksom, Demailly, Păun, and Peternell in [8], where they also give the following alternative characterization of the cone of movable curve classes:

Definition 4.1.1. Let $\mu : X' \to X$ be a projective, birational morphism. A class $\gamma \in \overline{\mathrm{NE}}(X)$ is called *movable* if there exists a representative one-cycle C, $[C] = \gamma$, and ample divisors $A_1, \ldots, A_{\dim(X)-1}$ on X' such that

$$\mu_*(A_1 \cdot \ldots \cdot A_{\dim(X)-1}) = C.$$

The closure of the cone generated by movable classes in $\overline{\mathrm{NE}}(X)$ is called the *movable cone*, and is denoted $\overline{\mathrm{Mov}}(X)$.

Both formulations involve non-trivial existence statements: in the first, to see that a curve C is movable, we must prove the existence of a covering family to which it belongs, and in the second, we require knowledge about all possible projective, birational morphisms to the variety X. If, however, we consider only the identity morphism, we obtain a subcone of $\overline{\mathrm{Mov}}(X)$ called the *complete intersection cone*:

Definition 4.1.2. The *complete intersection* cone of X, denoted $\mathcal{CI}(X)$, is the closed cone generated by the classes of all smooth curves obtained as an intersection of $\dim(X)-1$ ample divisors on X.

Were there actual equality $\mathcal{CI}(X) = \overline{\mathrm{Mov}}(X)$, then we could characterize movable curves without having to first classify all birational morphisms to X. A disadvantage of working with the complete intersection cone, however, is the combinatorial complexity of $\mathcal{CI}(X)$, especially when the nef cone of X has a large number of extremal rays.

Example 4.1.3. Let X be a smooth projective surface. Then one-cycles and divisors coincide, so $\overline{\mathrm{Eff}}(X)^\vee = \mathrm{Nef}(X) = \mathcal{CI}(X)$, where the second equality follows from Kleiman's theorem, since, by definition $\mathcal{CI}(X)$ is the closure of the ample cone.

Example 4.1.4. Let $X = \mathbb{P}^n$, and let $H \subseteq \mathbb{P}^n$ be a hyperplane, and let $L \subseteq \mathbb{P}^n$ be a line. Then $N^1(\mathbb{P}^n)_\mathbb{R} = \langle [H] \rangle$ and $\overline{\mathrm{Eff}}(\mathbb{P}^n) = \mathrm{Nef}(\mathbb{P}^n) = \langle [H] \rangle_{\geq 0}$. Turning to one-cycles, $N_1(\mathbb{P}^n)_\mathbb{R} = \langle [L] \rangle$, and the dual of the one-dimensional cone $\overline{\mathrm{Eff}}(\mathbb{P}^n) = \langle [H] \rangle_{\geq 0}$ is $\overline{\mathrm{Mov}}(\mathbb{P}^n) = \langle [L] \rangle_{\geq 0}$, but since $[H]^{n-1} = [L]$, it follows that $\mathcal{CI}(\mathbb{P}^n) = \overline{\mathrm{Mov}}(\mathbb{P}^n)$.

A further example will be given in Appendix 4.5, where we calculate the complete intersection of $\mathbb{P}^3$ blown up at a point. Peternell has calculated an example of a smooth projective threefold for which the containment of the complete intersection cone in the movable cone is strict ([51]), but one can ask if there are natural families of varieties for which these cones coincide. Two obvious testing grounds are toric varieties and moduli spaces of stable pointed rational curves, since the intersection theory on these varieties is well-understood. A connection between these two families is the Kapranov blow-up construction. As originally formulated in [36], $\overline{M}_{0,n}$ is constructed by a series of toric blow-ups of $\mathbb{P}^{n-3}$, culminating in the permutohedral or Losev-Manin moduli space $\overline{L}_{n-2}$, followed by (for $n \geq 5$) additional blow-ups along non-torus-invariant centers (see also Section 3.3). Example 4.1.4 can be taken as the base case of a progression of varieties obtained by successive Kapranov-like blow-ups. More specifically, setting $X_0 = \mathbb{P}^3$, the next variety we take to be the blow-up of $\mathbb{P}^3$ along a torus invariant point, labeling the resulting variety X_1. We define X_2 to be the blow-up of $\mathbb{P}^3$ along two torus-invariant points, and the proper transform of the line generated by the points. In general, for $1 \leq r \leq 5$, we blow-up r points of $\mathbb{P}^3$ in general linear position, and then the proper transforms of the $\binom{r}{2}$ lines generated by the r points. For $r \leq 4$, the centers of the blow-ups can be chosen to be torus-invariant. Note also that X_4 is the Losev-Manin moduli space $\overline{L}_4$ (see Section 3.2), while X_5 is $\overline{M}_{0,6}$. The complete intersection and movable cones of the first few varieties X_r can be computed easily to show that these cones coincide, but there is little reason to expect this equality of cones to be preserved under increasing blow-ups. One warning example in this direction is $\mathbb{P}^2$ blown up at ten or more very general points, whose closed cone of curves is not even finitely generated (see Section 1.5.D of [45]).

The main result of this chapter is that the inclusion of the complete intersection cone in the movable cone is strict for $\overline{M}_{0,6}$, but for the toric varieties X_r leading up to and including $\overline{L}_4$, the two cones coincide. In other words, the containment of these cones becomes strict when we leave the toric world in the Kapranov construction of $\overline{M}_{0,6}$.

Theorem 4.1.5. *For $\overline{M}_{0,6}$, there is a strict inclusion $\mathcal{CI}(\overline{M}_{0,6}) \subsetneq \overline{\mathrm{Mov}}(\overline{M}_{0,6})$, while for the toric varieties X_r, $1 \leq r \leq 4$, equality holds: $\mathcal{CI}(X_r) = \overline{\mathrm{Mov}}(X_r)$.*

We prove this theorem by reinterpreting the complete intersection cone in combinatorial terms (see Definition 4.4.1 and Lemma 4.4.2). Since the nef and pseudoeffective cones of $\overline{M}_{0,6}$ and $\overline{L}_4$ are finitely generated, it follows by this reinterpretation that the equality or inequality of the moving and complete intersection cones can be established by an algorithm that requires as input the extremal rays of the nef and effective cones of divisors, plus intersection products of divisors (see Section 4.4).

That the complete intersection and movable cones coincide for the toric blow-ups of Theorem 4.1.5 might give some hope that these cones coincide for smooth projective toric varieties. It turns out, however, that even for a toric blow-up of projective space, the complete intersection cone need not equal the movable cone. In Example 4.4.5, we consider the blow-up of $\mathbb{P}^3$ along two intersecting torus-invariant lines, and show that the complete intersection cone is strictly contained in the movable cone.

The remainder of this chapter is organized as follows. We begin with some generalities on the pseudoeffective and nef cones of divisors, as well as the the closed cones of curves, for $\overline{M}_{0,n}$ and the other blow-ups X_r in Section 4.2. In Section 4.3, we give explicit descriptions of the nef and movable cones of the X_r via inequalities, plus we calculate all possible intersections of pairs of divisor classes on these varieties (the calculations for parts of this section are found in Appendix 4.5). Lastly, in Section 4.4, we establish a combinatorial definition of the complete intersection cone, and describe the algorithm that proves Theorem 4.1.5.

4.2 Background on cones of divisors and curves

Except for small n, little is known about the pseudoeffective cone of divisors or the closed cone of curves for $\overline{M}_{0,n}$. The pseudoeffective cone of $\overline{M}_{0,n}$ (and hence, by duality, the movable cone of curves) is known to be finitely generated only for $n \leq 6$ ([33] and [10]), while finite-generation of the closed cone of curves of $\overline{M}_{0,n}$ (and hence, by duality, the cone of nef divisors) has been proven for $n \leq 7$ ([39] and Chapter 2).

For $\overline{M}_{0,6}$, the closed cone of curves is generated by classes of F-curves (see Definition 1.2.7), while the pseudoeffective cone of $\overline{M}_{0,6}$ is generated by the boundary divisors Δ_J, and the *Keel-Vermeire* divisors (see [55]). The latter are divisors obtained as pull-backs of the hyperelliptic locus in $\overline{M}_3$ under the morphism identifying the marked points in each of the pairs (a,b), (c,d), and (e,f), where $\{a,b,c,d,e,f\} = \{1,2,3,4,5,6\}$. In the Kapranov blow-up description of $\overline{M}_{0,6}$ (see Section 1.3.3), Keel-Vermeire divisors are the pull-backs under the blow-up morphism $t_6 : \overline{M}_{0,6} \to \mathbb{P}^3$ of the unique quadric surface containing points $l_1, \ldots, l_5$, and the lines l_{ac}, l_{ad}, l_{bc} and l_{bd}. Taking $(a,b,c,d) = (1,2,3,4)$, in the Kapranov basis, the Keel-Vermeire divisor $Q_{(12)(34)(56)}$ has class

$$[Q_{(12)(34)(56)}] = 2[H] - \sum_{i=1}^{5}[E_i] - [E_{13}] - [E_{14}] - [E_{23}] - [E_{24}], \tag{4.2.1}$$

with the remaining fourteen Keel-Vermeire divisors arising from different choices of a, b, c, and d (see [55], [33], and [10] for further details).

For toric varieties, however, each of the cones described above is finitely generated, and admits an explicit description (sometimes more than one) in combinatorial terms. The starting point for understanding the various cones of a toric variety X_Σ of dimension d is the Orbit-Cone correspondence (see Chapter 3, Theorem 3.2.2). Recalling that $\Sigma(k)$ denotes the k-dimensional cones of the fan Σ, and $V(\sigma)$ is the codimension k subvariety corresponding to $\sigma \in \Sigma(k)$, we have the following descriptions of $\overline{\mathrm{Eff}}(X_\Sigma)$ and $\overline{\mathrm{NE}}(X_\Sigma)$:

Proposition 4.2.1. *For a complete toric variety X_Σ,*

$$\overline{\mathrm{Eff}}(X_\Sigma) = \langle [V(\rho)] : \rho \in \Sigma(1) \rangle_{\geq 0},$$

while

$$\overline{\mathrm{NE}}(X_\Sigma) = \langle [V(\tau)] : \tau \in \Sigma(d-1) \rangle_{\geq 0}.$$

See [22] or [13] for proofs. The duals of these cones, $\overline{\mathrm{Mov}}(X_\Sigma)$ and $\mathrm{Nef}(X_\Sigma)$, can be calculated by the combinatorics of the defining fans. We will see examples of these calculations in Example 4.4.5 and in Appendix 4.5.

4.3 Calculating nef and movable cones

The proof of Theorem 4.1.5 involves comparing intersections of pairs of nef divisors on X_r with movable classes, which are defined by having non-negative intersections with all effective divisors on X_r. In this section, we give defining inequalities for the nef and movable cones, and calculate intersections of pairs of divisors on X_r, thus providing the input data for the algorithmic proof of Theorem 4.1.5 to be described in Section 4.4.

As the varieties X_r are related by Kapranov-like blow-ups, we will perform all calculations in the corresponding bases for $N^1(X_r)_\mathbb{R}$ and $N_1(X_r)_\mathbb{R}$.

Definition 4.3.1. Let X_r be the composition of the blow-ups of r general points in $\mathbb{P}^3$, $1 \leq r \leq 5$, followed by the blow-ups of the proper-transforms of the $\binom{r}{2}$ lines of $\mathbb{P}^3$ spanned by the r points.

Let H be the pullback of a general hyperplane, let $E_1, \ldots, E_r$ be the exceptional divisors obtained by blowing up the points, and let $E_{12}, \ldots, E_{r-1\,r}$ be the proper transforms of the exceptional divisors obtained by blowing up the lines.

The *Kapranov basis* of $N^1(X_r)_{\mathbb{R}}$ is

$$\{[H], [E_1], \ldots, [E_r], [E_{12}], \ldots, [E_{r-1\,r}]\},$$

and the *dual Kapranov basis* of $N_1(X_r)_{\mathbb{R}}$ is denoted

$$\{[H]^\vee, [E_1]^\vee, \ldots, [E_r]^\vee, [E_{12}]^\vee, \ldots, [E_{r-1\,r}]^\vee\}.$$

We will abuse notation and not introduce additional superscripts to distinguish the Kapranov bases from the different X_r. The basis intended will be clear from context. In Proposition 4.3.5 we will identify the elements of the Kapranov dual basis with intersections of elements of the Kapranov basis.

We recall next the dictionary between boundary divisors and the Kapranov basis for $\overline{M}_{0,6}$ as described in Section 1.3.3:

$$\begin{aligned} &\Delta_{i6} = E_i, \text{ for } 1 \leq i \leq 5, \\ &\Delta_{jk6} = E_{jk}, \text{ for } 1 \leq j < k \leq 5, \\ &[\Delta_{ab}] = [H] - \Big(\sum_{J \subsetneq \{a,b,6\}^c} [E_J] \Big), \end{aligned} \tag{4.3.1}$$

where $a, b \in \{1, \ldots, 5\}$. For example, setting $\{a, b\} = \{1, 3\}$ in the last type of equality above gives

$$\begin{aligned} [\Delta_{13}] &= [H] - [E_2] - [E_4] - [E_5] - [E_{24}] - [E_{25}] - [E_{45}] \\ &= [H] - [\Delta_{26}] - [\Delta_{46}] - [\Delta_{56}] - [\Delta_{246}] - [\Delta_{256}] - [\Delta_{456}]. \end{aligned}$$

Proposition 4.3.2. *Let $[D] = d_h[H] + \sum_{i=1}^{5} d_i[E_i] + \sum_{1 \leq j < k \leq 5} d_{jk}[E_{jk}]$ be an arbitrary divisor class in $\overline{M}_{0,6}$. The cone of nef divisors is determined by the inequalities*

$$\begin{cases} -d_{ij} \geq 0, & \text{for } 1 \leq i < j \leq 5, \\ d_h + d_i + d_j - d_{ij} \geq 0, & \text{for } 1 \leq i < j \leq 5, \\ -d_i + d_{ij} + d_{ik} \geq 0, & \text{for } 1 \leq i, j, k \leq 5, j < k, i \notin \{j, k\}, \\ d_h + d_i + d_{jk} + d_{lm} \geq 0, & \text{for } \{i, j, k, l, m\} = \{1, \ldots, 5\}, j < k, l < m. \end{cases}$$

Proof. These inequalities will follow by intersecting the divisor class $[D]$ with all F-curves using the intersection theory as developed in Section 1.3. Since the F-curve classes generate $\overline{\mathrm{NE}}(\overline{M}_{0,6})$ (see [39] or Section 2.3), the resulting inequalities will define $\mathrm{Nef}(\overline{M}_{0,6})$. Recall from Section 1.3.1 that every F-curve determines a partition of $\{1, \ldots, n\}$ into four subsets, $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, and that the numerical equivalence class of F is uniquely determined by this partition. We write the class of F as $F(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, and mean by the *type* of F the (unordered) four-tuple given by the cardinalities of the elements of the partition, $(|\mathcal{A}| : |\mathcal{B}| : |\mathcal{C}| : |\mathcal{D}|)$. Recall from Definition 1.2.9 that the *spine* of F is the unique component of the general member of F having four special points, while a *tail* of F is any connected component in the complement of the spine in the general member of F. Finally, we will make use of the identification $\psi_6 = [H]$ discussed in Section 1.3.3.

First we intersect the F-nef divisor $[D]$ with F-curves of type $(1 : 1 : 1 : 3)$ with the marked point 6 on the tail of F. For concreteness, suppose that the class of F is $F(1, 2, 3, \substack{4\\5\\6})$. Proposition 1.3.8 gives the equality

$$[H] \cdot F(1, 2, 3, \substack{4\\5\\6}) = \psi_6 \cdot F(1, 2, 3, \substack{4\\5\\6}) = 0,$$

since the sixth marked point is strictly contained in the last partition. Applying Proposition 1.3.1 to the divisor classes $[E_i] = [\Delta_{i6}]$ and $[E_{jk}] = [\Delta_{jk6}]$ yields $[E_i] \cdot F(1, 2, 3, \substack{4\\5\\6}) = 0$, and $[E_{jk}] \cdot F(1, 2, 3, \substack{4\\5\\6}) = 0$ unless $\{j, k\} = \{4, 5\}$, in which case the intersection number is -1. Hence

$$[D] \cdot F(1, 2, 3, \substack{4\\5\\6}) = -d_{45} \geq 0.$$

By symmetry, varying the marked points on F-curves of type $(1 : 1 : 1 : 3)$ with 6 on the tail thus gives the first inequality of the proposition.

Next we consider F-curves of type $(1 : 1 : 1 : 3)$ with the sixth marked point on the spine. By symmetry, it suffices to consider the F-curve with class $F(4, 5, 6, \substack{1\\2\\3})$. Propositions 1.3.8 and 1.3.1 imply that

$$\begin{aligned}
&[H] \cdot F(4, 5, 6, \substack{1\\2\\3}) = 1, \\
&[E_4] \cdot F(4, 5, 6, \substack{1\\2\\3}) = [\Delta_{46}] \cdot F(4, 5, 6, \substack{1\\2\\3}) = 1, \\
&[E_5] \cdot F(4, 5, 6, \substack{1\\2\\3}) = [\Delta_{56}] \cdot F(4, 5, 6, \substack{1\\2\\3}) = 1, \\
&[E_{45}] \cdot F(4, 5, 6, \substack{1\\2\\3}) = [\Delta_{456}] \cdot F(4, 5, 6, \substack{1\\2\\3}) = -1,
\end{aligned}$$

with all other intersections with members of the Kapranov basis equal to zero. It follows that

$$[D] \cdot F(4, 5, 6, \substack{1\\2\\3}) = d_h + d_4 + d_5 - d_{45} \geq 0.$$

Varying the distribution of the marked points (keeping 6 on the spine), we obtain $d_h + d_i + d_j - d_{ij} \geq 0$.

It remains to consider intersections with F-curves of type $(1 : 1 : 2 : 2)$. Assume first F is an F-curve with the marked point 6 is on a tail with class, for example, $F(1, 2, \substack{3\\4}, \substack{5\\6})$. Again using Propositions 1.3.8 and 1.3.1, the intersections with the elements of the

Kapranov basis are

$$\begin{aligned}
[E_5] \cdot F(1,2,\tfrac{3}{4},\tfrac{5}{6}) &= [\Delta_{56}] \cdot F(1,2,\tfrac{3}{4},\tfrac{5}{6}) = -1,\\
[E_{15}] \cdot F(1,2,\tfrac{3}{4},\tfrac{5}{6}) &= [\Delta_{156}] \cdot F(1,2,\tfrac{3}{4},\tfrac{5}{6}) = 1,\\
[E_{25}] \cdot F(1,2,\tfrac{3}{4},\tfrac{5}{6}) &= [\Delta_{256}] \cdot F(1,2,\tfrac{3}{4},\tfrac{5}{6}) = 1,
\end{aligned}$$

with all other intersections equal to zero. We obtain

$$[D] \cdot F(1,2,\tfrac{3}{4},\tfrac{5}{6}) = -d_5 + d_{15} + d_{25} \geq 0,$$

and therefore, by symmetry, $-d_i + d_{ij} + d_{ik} \geq 0$.

The final case to consider is F-curves of type $(1:1:2:2)$ with the sixth marked point on the spine. Taking the F-curve class $F(5,6,\frac{1}{2},\frac{3}{4})$, the intersection numbers are, as above,

$$\begin{aligned}
&[H] \cdot F(5,6,\tfrac{1}{2},\tfrac{3}{4}) = 1,\\
&[E_5] \cdot F(5,6,\tfrac{1}{2},\tfrac{3}{4}) = [\Delta_{56}] \cdot F(5,6,\tfrac{1}{2},\tfrac{3}{4}) = 1,\\
&[E_{12}] \cdot F(5,6,\tfrac{1}{2},\tfrac{3}{4}) = [\Delta_{126}] \cdot F(5,6,\tfrac{1}{2},\tfrac{3}{4}) = 1,\\
&[E_{34}] \cdot F(5,6,\tfrac{1}{2},\tfrac{3}{4}) = [\Delta_{346}] \cdot F(5,6,\tfrac{1}{2},\tfrac{3}{4}) = 1,
\end{aligned}$$

while all other intersection numbers with elements of the Kapranov basis are zero. It follows that

$$[D] \cdot F(5,6,\tfrac{1}{2},\tfrac{3}{4}) = d_h + d_5 + d_{12} + d_{34} \geq 0, \tag{4.3.2}$$

which, by varying the placement of the first through fifth marked point in the partitions gives the inequality $d_h + d_i + d_{jk} + d_{lm} \geq 0$. The above four cases exhaust all possible partitions corresponding to F-curves in $\overline{M}_{0,6}$, so these inequalities define the nef cone of $\overline{M}_{0,6}$. □

To calculate the movable cone of curves, $\overline{\mathrm{Mov}}(\overline{M}_{0,6})$, we consider intersections of one-cycles with the generators of $\overline{\mathrm{Eff}}(\overline{M}_{0,6})$, that is, with all boundary divisors Δ_J and the Keel-Vermeire divisors $Q_{(ab)(cd)(e6)}$.

Proposition 4.3.3. *Let $[C] = c_h[H]^\vee + \sum_{i=1}^5 c_i[E_i]^\vee + \sum_{1\leq j<k\leq 5} c_{jk}[E_{jk}]^\vee$ be an arbitrary one-cycle class in $N_1(\overline{M}_{0,6})$. The cone of movable curves in $\overline{M}_{0,6}$ is determined by the inequalities*

$$\begin{cases}
c_i \geq 0, & \text{for } i = 1,\dots,5,\\
c_{jk} \geq 0, & \text{for } 1 \leq j < k \leq 5,\\
c_h - c_i - c_j - c_k - c_{ij} - c_{ik} - c_{jk} \geq 0, & \text{for } 1 \leq i < j < k \leq 5,\\
2c_h - \sum_{i=1}^5 c_i - c_{jl} - c_{kl} - c_{jm} - c_{km} \geq 0, & \text{for } j,k,l,m \in \{1,\dots,5\} \text{ distinct.}
\end{cases}$$

Proof. Since we represent $[C] \in N_1(\overline{M}_{0,6})_{\mathbb{R}}$ with respect to the dual Kapranov basis, by duality these inequalities can just be read off of the coordinates of the generators for $\overline{\mathrm{Eff}}(\overline{M}_{0,6})$ expressed in the Kapranov basis for $N^1(\overline{M}_{0,6})_{\mathbb{R}}$. Specifically, a one-cycle class $[C] = c_h[H]^\vee + \sum_{i=1}^5 c_i[E_i]^\vee + \sum_{1\le j<k\le 5} c_{jk}[E_{jk}]^\vee$ is in $\overline{\mathrm{Mov}}(\overline{M}_{0,6})$ if and only if $[C]$ intersects all generators of $\overline{\mathrm{Eff}}(\overline{M}_{0,6})$ non-negatively.

Assume that $[C] \in \overline{\mathrm{Mov}}(\overline{M}_{0,6})$, and consider first boundary generators $[\Delta_J]$ of the form $\Delta_{i6} = E_i$, $1 \le i \le 5$. By definition of the dual basis,

$$[C] \cdot [E_i] = c_i \ge 0,$$

thus giving the first set of inequalities of the proposition. Taking next boundary generators with representatives $\Delta_{jk6} = E_{jk}$, $1 \le j < k \le 5$ gives the inequality

$$[C] \cdot [E_{jk}] = c_{jk} \ge 0,$$

while the boundary generators $[\Delta_{ab}]$ for $a \ne b$, $a, b \in \{1, \dots, 5\}$, give via Equation (4.3.1) the inequality

$$\begin{aligned}[C] \cdot [\Delta_{ab}] &= [C] \cdot ([H] - \sum_{J \subsetneq \{a,b,c\}^c} [E_J]) \\ &= c_h - c_i - c_j - c_k - c_{ij} - c_{ik} - c_{jk} \ge 0,\end{aligned}$$

where $\{a, b, i, j, k\} = \{1, \dots, 5\}$.

The only remaining generators of $\overline{\mathrm{Eff}}(\overline{M}_{0,6})$ are the classes of Keel-Vermeire divisors. The coordinates of the class of the divisor $Q_{(12)(34)(56)}$, for example, are given in Equation (4.2.1), resulting in the inequality

$$[C] \cdot [Q_{(12)(34)(56)}] = 2c_h - \sum_{i=1}^{5} c_i - c_{13} - c_{23} - c_{14} - c_{24}.$$

Intersecting $[C]$ with the remaining fourteen Keel-Vermeire divisors gives the final set of inequalities of the proposition. □

The remaining varieties X_r, $1 \le r \le 4$ are toric varieties, so the defining inequalities for $\mathrm{Nef}(X_r)$ and $\overline{\mathrm{Mov}}(X_r)$ can be calculated from the defining fans. We perform these calculations in Appendix 4.5, and record the resulting inequalities here.

Proposition 4.3.4. *For $r = 1, \dots, 4$, the defining inequalities for $\mathrm{Nef}(X_r)$ and $\overline{\mathrm{Mov}}(X_r)$ are as follows:*

(i) $[D] = d_h[H] + d_1[E_1]$ *is in* $\mathrm{Nef}(X_1)$ *if and only if*

$$\begin{cases} d_h + d_1 \ge 0, \\ -d_1 \ge 0; \end{cases}$$

while $[C] = d_h[H]^\vee + c_1[E_1]^\vee$ *is in* $\overline{\mathrm{Mov}}(X_1)$ *if and only if*

$$\begin{cases} c_1 \geq 0, \\ c_h - c_1 \geq 0. \end{cases}$$

(ii) $[D] = d_h[H] + d_1[E_1] + d_2[E_2] + d_{12}[E_{12}]$ *is in* $\mathrm{Nef}(X_2)$ *if and only if*

$$\begin{cases} -d_i + d_{12} \geq 0, & \textit{for } i = 1, 2, \\ d_h + d_1 + d_2 - d_{12} \geq 0, & \\ -d_{12} \geq 0; & \end{cases}$$

while $[C] = c_h[H]^\vee + c_1[E_1]^\vee + c_2[E_2]^\vee + c_{12}[E_{12}]^\vee$ *is in* $\overline{\mathrm{Mov}}(X_2)$ *if and only if*

$$\begin{cases} c_i \geq 0, & \textit{for } i = 1, 2 \\ c_{12} \geq 0, & \\ c_h - c_1 - c_2 - c_{12} \geq 0. & \end{cases}$$

(iii) $[D] = d_h[H] + d_1[E_1] + d_2[E_2] + d_3[E_3] + d_{12}[E_{12}] + d_{13}[E_{13}] + d_{23}[E_{23}]$ *is in* $\mathrm{Nef}(X_3)$ *if and only if*

$$\begin{cases} -d_i + d_{ij} + d_{ik} \geq 0, & \textit{for distinct } 1 \leq i, j, k \leq 3, \\ d_h + d_i + d_j - d_{ij} \geq 0, & \textit{for } 1 \leq i < j \leq 3, \\ -d_{ij} \geq 0, & \textit{for } 1 \leq i < j \leq 3, \end{cases}$$

while $[C] = c_h[H]^\vee + c_1[E_1]^\vee + c_2[E_2]^\vee + c_3[E_3]^\vee + c_{12}[E_{12}]^\vee + c_{13}[E_{13}]^\vee + c_{23}[E_{23}]^\vee$ *is in* $\overline{\mathrm{Mov}}(X_3)$ *if and only if*

$$\begin{cases} c_i \geq 0, & \textit{for } 1 \leq i \leq 3, \\ c_{ij} \geq 0, & \textit{for } 1 \leq i < j \leq 3, \\ c_h - c_i - c_j - c_{ij} \geq 0, & \textit{for } 1 \leq i < j \leq 3. \end{cases}$$

(iv) $[D] = d_h[H^{lm}] + \sum_{i=1}^4 d_i[E_i^{lm}] + \sum_{1 \leq j < k \leq 4} d_{jk}[E_{jk}^{lm}]$ *is in* $\mathrm{Nef}(\overline{L}_4)$ *if and only if*

$$\begin{cases} -d_i + d_{ij} + d_{ik} \geq 0, & \textit{for distinct } 1 \leq i, j, k \leq 4, \\ -d_{ij} \geq 0, & \textit{for } 1 \leq i < j \leq 4, \\ d_h + d_i + d_j - d_{ij} \geq 0, & \textit{for } 1 \leq i < j \leq 4, \end{cases}$$

while $[C] = c_h[H^{lm}]^\vee + \sum_{i=1}^4 c_i[E_i^{lm}]^\vee + \sum_{1 \leq j < k \leq 4} c_{jk}[E_{jk}^{lm}]^\vee$ *is in* $\overline{\mathrm{Mov}}(\overline{L}_4)$ *if and only if*

$$\begin{cases} c_i \geq 0, & \textit{for } 1 \leq i \leq 4, \\ c_{jk} \geq 0, & \textit{for } 1 \leq j < k \leq 4, \\ c_h - c_i - c_j - c_k - c_{ij} - c_{ik} - c_{jk} \geq 0, & \textit{for } 1 \leq i < j < k \leq 4. \end{cases}$$

We will give an alternate definition of the complete intersection cone in Section 4.4 involving intersections of nef divisors, so we next write the remaining intersections of elements of the Kapranov basis for $N^1(\overline{M}_{0,6})_{\mathbb{R}}$ in terms of the dual Kapranov basis.

Proposition 4.3.5. *The intersections of elements of the Kapranov basis for $\overline{M}_{0,6}$, in terms of the Kapranov dual basis, are, for distinct $i, j, k \in \{1, \dots, 5\}$,*

$$\begin{aligned}
&[H]^2 = [H]^\vee, \\
&[H] \cdot [E_i] = 0, \\
&[H] \cdot [E_{jk}] = [E_j] \cdot [E_{jk}] = [E_k] \cdot [E_{jk}] = -[E_{jk}]^\vee, \\
&[E_i]^2 = [E_i]^\vee, \\
&[E_i] \cdot [E_j] = 0, \\
&[E_i] \cdot [E_{jk}] = 0, \\
&[E_{jk}]^2 = 2[E_{jk}]^\vee - [H]^\vee - [E_j]^\vee - [E_k]^\vee, \\
&[E_{ab}] \cdot [E_{cd}] = 0,
\end{aligned}$$

where $a, b, c, d \in \{1, \dots, 5\}$ satisfy $a \neq b$, $c \neq d$, and $\{a, b\} \neq \{c, d\}$.

Proof. Some of these equalities can be found in [10], Section 9, and [33], but for completeness we (re)prove all of the equalities above. We split the proof into two stages, first proving the stated equalities among the various intersections of pairs of divisor classes, and secondly proving the equalities with elements of the dual Kapranov basis.

The equality $[H]\cdot[E_i] = 0$ follows from the projection formula: recall that $t_6 : \overline{M}_{0,6} \to \mathbb{P}^3$ is the composition of blow-downs in the Kapranov construction, with the sixth point chosen as the moving point. If $H' \subseteq \mathbb{P}^3$ is a generic plane, then by definition, $H = t_6^*(H')$. By the projection formula,

$$[H] \cdot [E_i] = [t_6^*(H')] \cdot [E_i] = [H'] \cdot [(t_6)_*(E_i)] = 0,$$

since t_6 contracts E_i.

The third equality results from combining the dictionary between boundary divisors and the Kapranov basis with Keel's relations. By symmetry, it suffices to prove

$$[H] \cdot [E_{12}] = [E_1] \cdot [E_{12}] = [E_2] \cdot [E_{12}].$$

By the third of Equations (4.3.1) plus the equality $E_J = \Delta_{J \cup \{6\}}$, we can rewrite the class of H as

$$[H] = [\Delta_{13}] + [\Delta_{26}] + [\Delta_{46}] + [\Delta_{56}] + [\Delta_{246}] + [\Delta_{256}] + [\Delta_{456}].$$

Recall from Section 1.3.2 that $[\Delta_{J_1}] \cdot [\Delta_{J_2}] = 0$ unless one of the following holds:

$$J_1 \subseteq J_2, \quad J_2 \subseteq J_1, \quad J_1 \subseteq J_2^c, \quad J_1^c \subseteq J_2. \tag{4.3.3}$$

Therefore all intersections of the right hand side of the above expression for $[H]$ with

$E_{12} = \Delta_{126}$ are zero except for $[\Delta_{26}] \cdot [\Delta_{126}] = [E_2] \cdot [E_{12}]$, so $[H] \cdot [E_{12}] = [E_2] \cdot [E_{12}]$.

If we rewrite the class of H instead as

$$[H] = [\Delta_{23}] + [\Delta_{16}] + [\Delta_{46}] + [\Delta_{56}] + [\Delta_{146}] + [\Delta_{156}] + [\Delta_{456}],$$

then as before, all intersections of $E_{12} = \Delta_{126}$ with the right hand side of the above expression for $[H]$ vanish except $[\Delta_{16}] \cdot [\Delta_{126}]$, thus giving $[H] \cdot [E_{12}] = [E_1] \cdot [E_{12}]$.

The equality $[E_i] \cdot [E_j] = 0$ for $i \neq j$ follows from observing that the index sets of $E_i = \Delta_{i6}$ and $E_j = \Delta_{j6}$ do not satisfy any of the conditions in Equation (4.3.3). We likewise obtain $[E_i] \cdot [E_{jk}] = [\Delta_{i6}] \cdot [\Delta_{jk6}] = 0$ for i, j, k distinct.

For $a, b, c, d \in \{1, \ldots, 5\}$ such that $a \neq b$, $c \neq d$, and $\{a, b\} \neq \{c, d\}$, the equality $[E_{ab}] \cdot [E_{cd}] = [\Delta_{ab6}] \cdot [\Delta_{cd6}] = 0$ follows similarly by noting that the index sets $\{a, b, 6\}$ and $\{c, d, 6\}$ satisfy none of the inclusions from Equation (4.3.3).

It remains to show how these intersections are expressed in the Kapranov basis for $N_1(\overline{M}_{0,6})_{\mathbb{R}}$. The equality $[H]^2 = [H]^\vee$ can be obtained from the projection formula as follows. Letting again $H' \subseteq \mathbb{P}^3$ be a generic plane, for any exceptional divisor E_J,

$$\begin{aligned} [H]^2 \cdot [E_J] &= [t_6^*(H')]^2 \cdot [E_J] \\ &= t_6^*([H']^2) \cdot [E_J] \\ &= [H']^2 \cdot (t_6)_*[E_J] = 0, \end{aligned}$$

where we have used that t_6^* is a homomorphism of Chow rings (see Fulton [23], Section 8.3). Similarly, $[H]^3 = [t_6^*(H')]^3 = [H']^3 = 1$, hence $[H]^2 = [H]^\vee$.

For the equality $[E_i]^2 = [E_i]^\vee$, first note that $[E_i]^2 \cdot [H] = 0$, since we have already shown that $[E_i] \cdot [H] = 0$. The remaining intersections can be calculated with help from the Keel relations of Equation (1.3.1). By symmetry, we only consider the case $[E_1] = [\Delta_{16}]$. We can rewrite $[\Delta_{16}]$ as

$$[\Delta_{16}] = -[\Delta_{23}] - [\Delta_{146}] - [\Delta_{156}] + [\Delta_{12}] + [\Delta_{36}] + [\Delta_{346}] + [\Delta_{356}],$$

where, in the notation of Equation (1.3.1) we have chosen $\{i, j, k, l\} = \{1, 2, 3, 6\}$. Keeping in mind the criteria of Equation (4.3.3), it follows that

$$\begin{aligned} [E_1]^2 &= [\Delta_{16}] \cdot (-[\Delta_{23}] - [\Delta_{146}] - [\Delta_{156}] + [\Delta_{12}] + [\Delta_{36}] + [\Delta_{346}] + [\Delta_{356}]) \\ &= -[\Delta_{16}] \cdot ([\Delta_{23} + [\Delta_{146}] + [\Delta_{156}]). \end{aligned}$$

Recalling Notation 1.3.3, we can express $[E_1]^2$ as the sum of F-curve classes,

$$[E_1]^2 = -F(4, 5, \tfrac{1}{6}, \tfrac{2}{3}) - F(2, 3, 5, \substack{1\\4\\6}) - F(2, 3, 4, \substack{1\\5\\6}).$$

It remains to show that this sum of F-curve classes has intersection number zero with all exceptional divisors except E_1, with which it has intersection number -1.

For the intersections of the exceptional divisors $E_J = \Delta_{J \cup \{6\}}$ with $[E_1]^2$, we use Proposition 1.3.1, which gives intersection numbers of F-curve classes with boundary divisors. Taking first boundary divisors Δ_{i6}, the only non-zero intersection of Δ_{i6} with one of the F-curve classes in the expression for $[E_1]^2$ is $[\Delta_{16}] \cdot F(4, 5, \substack{1\\6}, \substack{4\\5}) = -1$, hence $[E_1]^2 \cdot [\Delta_{i6}] = [E_1]^2 \cdot [E_i] = 1$ if $i = 1$, and is zero otherwise.

Finally, we intersect $[E_1]^2$ with $[E_{ij}] = [\Delta_{ij6}]$. Proposition 1.3.1 implies that $[\Delta_{ij6}] \cdot F(4, 5, \substack{1\\6}, \substack{2\\3}) = 1$ if $\{i, j\} = \{1, 4\}$ or $\{1, 5\}$, and is zero otherwise. Turning to the other F-curve classes in the sum, $[\Delta_{ij6}] \cdot F(2, 3, 5, \substack{1\\4\\6}) = -1$ if $\{i, j\} = \{1, 4\}$, and is zero otherwise, while $[\Delta_{ij6}] \cdot F(2, 3, 4, \substack{1\\5\\6}) = -1$ if $\{i, j\} = \{1, 5\}$, and equals zero otherwise. Combining it follows that $[E_1]^2 \cdot [E_{ij}] = 0$ for all i, j, as desired, showing that $[E_1]^2 = [E_1]^\vee$.

To prove that $[H] \cdot [E_{ij}] = -[E_{ij}]^\vee$, by symmetry and the first stage of the proof, it is enough to show that $[E_1] \cdot [E_{12}] = -[E_{12}]^\vee$. In addition, the first part of the proof shows that the only possible non-zero intersections with elements of the Kapranov basis and $[E_1]\cdot[E_{12}]$ are with the exceptional divisor classes $[E_1]$ and $[E_{12}]$. We have just shown that $[E_1]^2\cdot[E_{12}] = 0$, so it remains to consider $([E_1]\cdot[E_{12}])\cdot[E_{12}]$. As an F-curve class, we have $[E_1]\cdot[E_{12}] = F(3, 4, 5, \substack{1\\2\\6})$, so by Proposition 1.3.1, $[E_1]\cdot[E_{12}]^2 = [\Delta_{126}]\cdot F(3, 4, 5, \substack{1\\2\\6}) = -1$.

To show that $[E_{jk}]^2 = 2[E_{jk}]^\vee - [H]^\vee - [E_j]^\vee - [E_k]^\vee$, it suffices to consider $[E_{12}]^2 = [\Delta_{126}]^2$, which is calculated in Example 1.3.5 to be the sum of F-curve classes

$$[\Delta_{126}]^2 = -F(3, 4, 5, \substack{1\\2\\6}) - F(1, 2, 6, \substack{3\\4\\5}).$$

Applying the equality $[H] = \psi_6$, Proposition 1.3.8, and Proposition 1.3.1 one final time gives

$$\begin{aligned}
&[\Delta_{126}]^2 \cdot [H] = -1,\\
&[\Delta_{126}]^2 \cdot [\Delta_{16}] = [\Delta_{126}]^2 \cdot [\Delta_{26}] = -1,\\
&[\Delta_{126}]^2 \cdot [\Delta_{126}] = -(-1 - 1) = 2,
\end{aligned}$$

with all other intersections with boundary divisors of the form $\Delta_{J\cup\{6\}}$ equal to zero.

□

In the remainder of this section, we give one example of how the Losev-Manin spaces encode part of the geometry of $\overline{M}_{0,n}$, by showing that extremal rays of the movable cone of $\overline{L}_{n-2}$ pull back to extremal rays of the movable cone of $\overline{M}_{0,n}$.

First, note that $f^* : N^1(\overline{L}_{n-2})_{\mathbb{R}} \to N^1(\overline{M}_{0,n})_{\mathbb{R}}$ restricts to maps of cones $f^* : \overline{\mathrm{Eff}}(\overline{L}_{n-2}) \to \overline{\mathrm{Eff}}(\overline{M}_{0,n})$, with boundary divisor classes on $\overline{L}_{n-2}$ pulling back to the boundary divisor classes on $\overline{M}_{0,n}$ (Corollary 3.3.14).

Proposition 4.3.6. *Let R be an extremal ray of $\overline{\mathrm{Mov}}(\overline{L}_{n-2})$. Then $f^*(R)$ is an extremal ray of $\overline{\mathrm{Mov}}(\overline{L}_{n-2})$.*

Proof. The proof of this proposition uses the specific form $f^* : N^1(\overline{L}_{n-2})_{\mathbb{R}} \to N^1(\overline{M}_{0,n})_{\mathbb{R}}$ takes on boundary divisors, namely, that class of boundary divisor Δ_J^{lm} of $\overline{L}_{n-2}$ pulls back to the class of Δ_J of $\overline{M}_{0,n}$ (see Corollary 3.3.14), as well as an elementary result about polyhedral cones.

First we must show that, for $\gamma \in \overline{\mathrm{Mov}}(\overline{L}_{n-2})$, the pulled-back class $f^*(\gamma)$ is movable in $\overline{M}_{0,n}$. Let D be an effective divisor on $\overline{M}_{0,n}$. By the projection formula, $f^*(\gamma)\cdot D = \gamma \cdot f_*(D) \geq 0$, since $f_*(D)$ is an effective divisor on $\overline{L}_{n-2}$.

For the basic result about polyhedral cones that we will use in the proof, let $\sigma \subsetneq V$ be a closed, convex, full-dimensional polyhedral cone in a finite-dimensional real vector space V, such that σ contains no lines. Recall that a ray $R \subseteq \sigma$ is *extremal* if for every decomposition $R = S_1 + S_2$, where S_1, S_2 are rays of σ, it follows that S_1 and S_2 are scalar multiples of R. Let $P_1^+, \dots P_m^+ \subseteq V$ be the defining half-spaces for σ with corresponding supporting hyperplanes $P_1, \dots, P_m$. In the dual space V^*, we define $\sigma^\vee = \{\phi \in V^* : \phi(v) \geq 0 \text{ for all } v \in \sigma\}$. We can pick functionals $\phi_1, \dots, \phi_m$ such that for all $i = 1, \dots, m$, $P_i^+ = \{v \in V : \phi_i(v) \geq 0\}$, and $P_i = \{v \in V : \phi_i(v) = 0)$, which we denote also by $\phi_i^\perp$. A *face* of σ is defined as the intersection of σ with a hyperplane $\phi^\perp$, where $\phi \in \sigma^\vee$. We will use of the following fact about the polyhedral cone σ: a ray $R \subseteq \sigma$ is an extremal ray if and only if R is a one-dimensional face of σ (see, e.g. [50], Proposition 7.2).

Turning again to the movable cone, note first that $\overline{\mathrm{Mov}}(\overline{L}_{n-2})$ is polyhedral and cannot contain a line, for if it did contain a line, then there would exist a non-zero one-cycle class γ' such that both $\gamma' \cdot D \geq 0$ and $(-\gamma') \cdot D \geq 0$ for all effective divisors D. But the pseudoeffective cone of $\overline{L}_{n-2}$ is full-dimensional (see section 5.1 [22]). It follows that $\gamma' = 0$, a contradiction. Hence the above fact implies that the extremal rays of $\overline{\mathrm{Mov}}(\overline{L}_{n-2})$ are precisely the one-dimensional faces of $\overline{\mathrm{Mov}}(\overline{L}_{n-2})$. A one-dimensional face of $\overline{\mathrm{Mov}}(\overline{L}_{n-2})$ is further equal to the intersections of all of the facets containing it (see 1.2 (6) of [22]). Note that these facets have the form $P_i \cap \overline{\mathrm{Mov}}(\overline{L}_{n-2})$, where the P_i are the supporting hyperplanes of $\overline{\mathrm{Mov}}(\overline{L}_{n-2})$. But by [8] coupled with Proposition 4.2.1, each supporting hyperplanes of $\overline{\mathrm{Mov}}(\overline{L}_{n-2})$ is of the form $P_J = \{\gamma \in N^1(\overline{L}_{n-2}) : \gamma \cdot \Delta_J^{lm} = 0, \Delta_J^{lm} \text{ a boundary divisor of } \overline{L}_{n-2}\}$, or, equivalently, $P_J = \phi_J^\perp$, where ϕ_J is the linear functional defined by intersecting a one-cycle with the divisor Δ_J^{lm}.

Let $\gamma \in \overline{\mathrm{Mov}}(\overline{L}_{n-2})$ be an extremal movable class (i.e. the ray generated by γ is extremal). We fix isomorphisms $N^1(\overline{L}_{n-2})_{\mathbb{R}} \cong \mathbb{R}^\rho$ and $N^1(\overline{M}_{0,n-2})_{\mathbb{R}}$ by choosing the respective Kapranov bases, namely, for $N^1(\overline{L}_{n-2})_{\mathbb{R}}$ we choose $\{[H^{lm}], [E_J^{lm}] : J \subseteq \{1, \dots, n-2\}, 1 \leq |J| \leq n-4\}$, and for $N^1(\overline{M}_{0,n})_{\mathbb{R}}$ we choose $\{[H], [E_J] : J \subseteq \{1, \dots, n-1\}, 1 \leq |J| n-4\}$. Then f^* give a decomposition via Corollary 3.3.10,

$$\begin{aligned} N^1(\overline{M}_{0,n})_{\mathbb{R}} &= f^*(N^1(\overline{L}_{n-2})_{\mathbb{R}}) \oplus \langle [E_J] : 1 \leq |J| \leq n-4, J \subseteq \{1, \dots, n-1\}, n-1 \in J \rangle \\ &= \langle [H], [E_J] : 1 \leq |J| \leq n-4, J \subseteq \{1, \dots, n-2\} \rangle \qquad (4.3.4) \\ &\qquad \oplus \langle [E_J] : 1 \leq |J| \leq n-4, J \subseteq \{1, \dots, n-1\}, n-1 \in J \rangle. \end{aligned}$$

To finish the proof, by the projection formula, $f^*(\gamma) \cdot [E_J] = \gamma \cdot f_*([E_J])$, but for J with $n-1 \in J$, $f_*([E_J]) = 0$, so for such J, $f^*(\gamma) \in \phi_J^\perp$. In $N^1(\overline{L}_{n-2})$ we can write the ray corresponding to γ as $\langle \gamma \rangle_{\geq 0} = \cap((\Delta_{J_i}^{lm})^\perp \cap \overline{\mathrm{Mov}}(\overline{L}_{n-2}))$ for some collection of boundary divisors $\Delta_{J_i}^{lm}$. Again, classes of boundary divisors pull back to classes of boundary divisors (and hence supporting hyperplanes of $\overline{\mathrm{Mov}}(\overline{M}_{0,n})$), so

$$\langle f^*(\gamma)\rangle_{\geq 0} \subseteq (\cap(\Delta_{J_i})^\perp \cap \overline{\mathrm{Mov}}(\overline{M}_{0,n})) \cap (\cap_{\substack{n-1\in J\\ 1\leq |J|\leq n-4}} (\Delta_{J\cup\{n\}})^\perp \cap \overline{\mathrm{Mov}}(\overline{M}_{0,n})),$$

but since the hyperplanes $\phi_J^\perp = (\Delta_{J\cup\{n\}})^\perp$, $n-1 \in J$, $1 \leq |J| \leq n-4$, are coordinate hyperplanes in $N^1(\overline{M}_{0,n})_{\mathbb{R}}$, by the decomposition of Equation (4.3.4) the dimension of the right side of the above inclusion is also one, forcing equality. Hence the ray generated by $f^*(\gamma)$ is a one-face, and therefore an extremal ray of $\overline{\mathrm{Mov}}(\overline{M}_{0,n})$. □

4.4 Complete intersection and movable curves in $\overline{M}_{0,6}$

In this section we describe the algorithm used to prove Theorem 4.1.5. We begin with a recasting of the complete intersection cone of a projective variety X with a finitely generated nef cone.

Definition 4.4.1. For X a projective variety of dimension d with a finitely generated nef cone, define $(\mathrm{Nef}(X))^{d-1} \subseteq N_1(X)$ as

$$(\mathrm{Nef}(X))^{d-1} = \langle R_1 \cdot \ldots \cdot R_{d-1} : \text{ each } R_i \text{ is an extremal ray of } \mathrm{Nef}(X)\rangle_{\geq 0}$$

Note that finite generation of $\mathrm{Nef}(X)$ implies that $(\mathrm{Nef}(X))^{d-1}$ is a closed cone.

Lemma 4.4.2. $\mathcal{CI}(X) = (\mathrm{Nef}(X))^{d-1}$.

Proof. To see that $(\mathrm{Nef}(X))^{d-1} \subseteq \mathcal{CI}(X)$, note first that every nef divisor is a limit of ample divisors (see [45], Section 1.4). Thus by multilinearity and continuity of the intersection product ([45], Section 1.1), the generators of $(\mathrm{Nef}(X))^{d-1}$ are in $\mathcal{CI}(X)$ since $\mathcal{CI}(X)$ is closed. By the convexity of $\mathcal{CI}(X)$, every finite positive sum of the generators of $(\mathrm{Nef}(X))^{d-1}$ is also contained in $\mathcal{CI}(X)$.

For the other inclusion, let $C = A_1 \cdot \ldots \cdot A_{d-1}$ be in the interior of $\mathcal{CI}(X)$. Then each A_i can be written as a non-negative $\mathbb{R}$-linear combination of extremal rays of $\mathrm{Nef}(X)$. Again, by multilinearity of the intersection product, the interior of $\mathcal{CI}(X)$ is contained in $(\mathrm{Nef}(X))^{d-1}$. Since $(\mathrm{Nef}(X))^{d-1}$ is closed, the result follows. □

This reformulation enables in principle an algorithmic proof of whether or not the containment $\mathcal{CI}(X) \subseteq \overline{\mathrm{Mov}}(X)$ is strict whenever $\mathrm{Nef}(X)$ and $\overline{\mathrm{Mov}}(X)$ are finitely generated.

Corollary/Algorithm 4.4.3. *If the nef cone of X is finitely generated, the cones $\mathcal{CI}(X)$ and $\overline{\mathrm{Mov}}(X)$ coincide if and only if every extremal ray of $\overline{\mathrm{Mov}}(X)$ is a non-trivial multiple of a generator of* $(\mathrm{Nef}(X))^{d-1}$.

We now describe the algorithm in detail for a projective three-fold; the extension of the algorithm to higher-dimensional varieties will be obvious. We assume for the algorithm that both $\mathrm{Nef}(X)$ (and hence $\mathcal{CI}(X)$) and $\overline{\mathrm{Mov}}(X)$ are finitely generated.

Let $R_1, \ldots, R_s$ be an enumeration of the extremal rays of $\overline{\mathrm{Mov}}(X)$, and let $N_1, \ldots, N_t$ be an enumeration of the extremal rays of $\mathrm{Nef}(X)$. Choose bases for $N^1(X)_{\mathbb{R}}$ and $N_1(X)_{\mathbb{R}}$, calling them $\mathcal{B}^1$ and $\mathcal{B}_1$, respectively. For $i \in \{1, \ldots, s\}$, stage i of the algorithm for X a projective three-fold begins by calculating the generators of $\mathcal{CI}(X)$, $C_{t_1,t_2} = N_{t_1} \cdot N_{t_2}$ in the basis $\mathcal{B}_1$ for all $1 \leq t_1 \leq t_2 \leq t$. If $C_{1,1}$ is a positive multiple of R_i, then we continue to the $(i+1)^{\mathrm{st}}$ stage of the algorithm. If not, we compare R_i with $C_{1,2}$. If R_i is a positive multiple of $C_{1,2}$, we continue to stage $i+1$. If not, we continue until either we find a generator C_{t_1,t_2} of $\mathcal{CI}(X)$ proportional to R_i, or until we have checked R_i against all generators C_{t_1,t_2}. If no generator of $\mathcal{CI}(X)$ is a positive multiple of R_i, then $R_i \in \overline{\mathrm{Mov}}(X) \setminus \mathcal{CI}(X)$. The two cones are equal if and only if the algorithm finds a matching generator of $\mathcal{CI}(X)$ for each extremal ray R_i, $i = 1, \ldots, s$.

Note that the algorithm involves recalculating all generators for $\mathcal{CI}(X)$ given by the equality $\mathcal{CI}(X) = (\mathrm{Nef}(X))^2$ for each extremal ray of $\overline{\mathrm{Mov}}(X)$. This apparent inefficiency is in practice preferable to storing every generator $N_{t_1} \cdot N_{t_2}$ in an array due to memory requirements and the computational time required to access elements in this array.

An implementation of the algorithm as a C++ program for X_r, $r = 2, \ldots, 5$, is available at www.math.hu-berlin.de/∼larsen. We obtain enumerations of the extremal rays of $\mathrm{Nef}(X)$ and $\overline{\mathrm{Mov}}(X)$ by inputting the inequalities into PORTA from Propositions 4.3.2 and 4.3.3 for $\overline{M}_{0,6}$, and from Proposition 4.3.4 for $\overline{L}_4$ and the other X_r. These PORTA files are also available at www.math.hu-berlin/∼larsen. Intersections of pairs of nef divisors are calculated according to Proposition 4.3.5 for $\overline{M}_{0,6}$, and as given in Appendix 4.5 for the other X_r. Running these programs yields:

Corollary 4.4.4. *$\mathcal{CI}(\overline{M}_{0,6}) \subsetneq \overline{\mathrm{Mov}}(\overline{M}_{0,6})$, while $\mathcal{CI}(X_r) = \overline{\mathrm{Mov}}(X_r)$ for $r = 1, \ldots, 4$.*

To illustrate the algorithm, we calculate $\mathcal{CI}(X_1)$ and $\overline{\mathrm{Mov}}(X_1)$ by hand in Appendix 4.5.

Combining with Proposition 4.3.6, and varying which marked points are chosen as poles for $\overline{L}_4$, we obtain an enumeration of extremal rays common to $\overline{\mathrm{Mov}}(\overline{M}_{0,6})$ and $\mathcal{CI}(\overline{M}_{0,6})$. This collection, however, does not give all common extremal rays: for example, the extremal ray

$$R = 6[H^\vee] + 2\sum_{i=1}^{4}[E_i^\vee] + [E_{15}^\vee] + [E_{25}^\vee] + [E_{35}^\vee],$$

and its symmetric analogues, is an extremal ray of both $\mathcal{CI}(\overline{M}_{0,6})$ and $\overline{\mathrm{Mov}}(\overline{M}_{0,6})$, but it is not the pull-back of an extremal ray from $\overline{\mathrm{Mov}}(\overline{L}_4)$, as can be seen by examining the PORTA file for $\overline{\mathrm{Mov}}(\overline{L}_4)$.

An obvious question to ask is whether the complete intersection and movable cones coincide for all smooth projective toric varieties. We conclude with an example of a toric blow-up of $\mathbb{P}^3$ for which the complete intersection cone is strictly contained in the movable cone.

Example 4.4.5. Let Y_2 be the toric variety obtained by blowing up $\mathbb{P}^3$ first in the line $V(z_1, z_3)$, followed by the blow-up of the proper-transform of the line $V(z_1, z_2)$, where

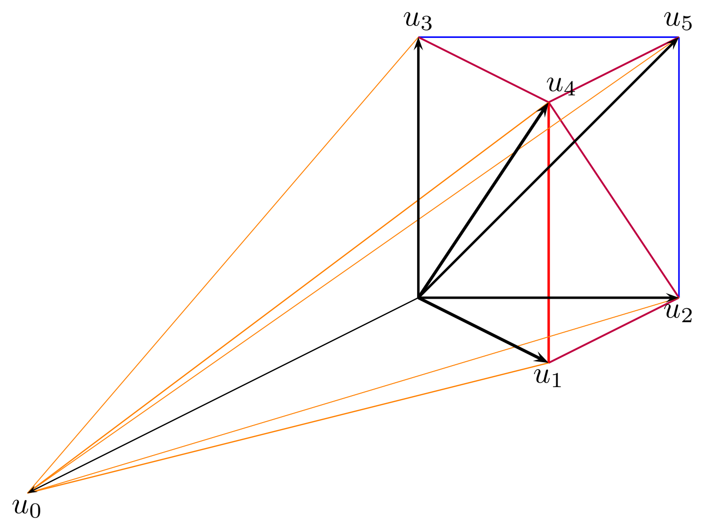


Figure 4.1: Fan of the toric variety from Example 4.4.5

$\mathbb{C}[z_0, z_1, z_2, z_3]$ is the homogeneous coordinate ring of $\mathbb{P}^3$. Y_2 is smooth and projective as a blow-up of a smooth, projective variety, and its fan Σ is depicted in Figure 4.1, with the other segments indicating the two-faces of the fan. The standard facts about divisor classes and intersection theory on toric varieties reviewed and used in this example can be found in [13], Chapters 4 and 6.

The primitive generators of the fan of Y_2 are

$$u_0 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, u_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, u_5 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

As usual, we label the torus-invariant divisors via the (primitive generators of the) rays of $\Sigma(1)$, so $N^1(Y_2)_{\mathbb{R}}$ is generated by the classes of the divisors $D_0, D_1, D_2, D_3, D_4, D_5$.

In general, the relations among the divisors D_ρ of a toric variety X_Σ are generated by the principal divisors

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, w_\rho \rangle D_\rho,$$

where $m \in M$, and w_ρ is a primitive generating vector for ρ (see [13], Section 4.1).

For Y_2, it suffices to consider the characters χ^m given by the three coordinate functions on the lattice $N \cong \mathbb{Z}^3$. Hence the relations among the classes of divisors D_i are generated by

$$
\begin{aligned}
&[D_1] + [D_4] - [D_0] = 0,\\
&[D_2] + [D_5] - [D_0] = 0,\\
&[D_3] + [D_4] + [D_5] - [D_0] = 0.
\end{aligned}
$$

We now perform the calculations of Algorithm 4.4.3 to compare $\mathcal{CI}(Y_2)$ and $\overline{\mathrm{Mov}}(Y_2)$. It is clear from the relations in $N^1(Y_2)_\mathbb{R}$ that the pseudoeffective cone is

$$\overline{\mathrm{Eff}}(Y_2) = \langle [D_3], [D_4], [D_5] \rangle_{\geq 0}.$$

We therefore choose the basis for $N^1(Y_2)_\mathbb{R}$ consisting of the classes of D_3, D_4, and D_5, while for $N_1(Y_2)_\mathbb{R}$ we take the corresponding dual basis. It follows that the movable cone is

$$\overline{\mathrm{Mov}}(Y_2) = \langle [D_3]^\vee, [D_4]^\vee, [D_5]^\vee \rangle_{\geq 0},$$

or, in coordinates, the non-negative orthant of $\mathbb{R}^3$.

As noted in Proposition 4.2.1, the closed cone of curves of Y_2 is generated by classes of orbit closures $V(\tau)$, $\tau \in \Sigma(2)$, and we will label them as $V(\tau) = C_{i,j}$, where i and j index the rays generating τ.

Writing an arbitrary divisor class as $[D] = d_3[D_3] + d_4[D_4] + d_5[D_5]$, the nef cone of Y_2 is defined by the inequalities

$$
\begin{aligned}
d_4 &= [C_{0,1}] \cdot [D] = [C_{1,2}] \cdot [D] = [C_{1,4}] \cdot [D] = [C_{2,5}] \cdot [D] \geq 0,\\
d_5 &= [C_{0,2}] \cdot [D] \geq 0,\\
-d_4 + d_5 &= [C_{2,4}] \cdot [D] \geq 0,\\
-d_3 + d_4 + d_5 &= [C_{0,3}] \cdot [D] = [C_{3,4}] \cdot [D] = [C_{3,5}] \cdot [D] \geq 0,\\
d_3 - d_5 &= [C_{0,5}] \cdot [D] = [C_{4,5}] \cdot [D] \geq 0,\\
d_3 - d_4 &= [C_{0,4}] \cdot [D] \geq 0.
\end{aligned}
$$

These intersections can be calculated via the geometry of the fan of Y_2 as follows. For a smooth, complete toric threefold X_Σ, each torus invariant curve C is the closure of the orbit of some $\tau = \langle \rho_i, \rho_j \rangle_{\geq 0}$, and, since the fan is complete (that is, its support is all of $\mathbb{R}^3$), there are precisely two cones $\sigma_a, \sigma_b \in \Sigma(3)$ containing τ. Since Σ is also smooth, the generators of all cones form part of a $\mathbb{Z}$-basis for the lattice N, hence there exists $\rho_a, \rho_b \in \Sigma(1)$ with $\sigma_a = \langle \rho_a, \rho_1, \rho_2 \rangle_{\geq 0}$, and $\sigma_b = \langle \rho_1, \rho_2, \rho_b \rangle_{\geq 0}$.

To intersect the curve $C = V(\tau)$ with divisors D_ρ, we consider two cases. If $\rho \notin \{\rho_a, \rho_1, \rho_2, \rho_b\}$, then C and D_ρ are disjoint, so $C \cdot D_\rho = 0$. For the remaining divisors, note that there is a non-trivial relation over $\mathbb{Z}$

$$\alpha\rho_a + c_1\rho_1 + c_2\rho_2 + \beta\rho_b = 0.$$

Since ρ_a and ρ_b are on opposite sides of τ, we may choose both α and β to be positive.

By the smoothness assumption, we may further take $\alpha = \beta = 1$ (see [13], Corollary 6.3.3, or [22], Section 5.1). The remaining intersections can now be read off of the linear dependence relation: $C \cdot D_{\rho_1} = c_1$ and $C \cdot D_{\rho_2} = c_2$. For proofs we refer to [13], Section 6.3, which treats the more general case of a simplicial fan Σ.

For example, to obtain the fourth inequality above, we intersect D with the curve $C_{0,3}$. Setting $C_{0,3} = V(\tau)$, with $\tau = \langle u_0, u_3 \rangle_{\geq 0}$, note that τ is contained precisely in the full-dimensional cones $\langle u_4, u_0, u_3 \rangle_{\geq 0}$ and $\langle u_0, u_3, u_5 \rangle_{\geq 0}$. We obtain from the coefficients of the linear dependence relation

$$(1)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + (1)\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

the intersection numbers $D_4 \cdot C_{0,3} = 1$, $D_0 \cdot C_{0,3} = 1$, $D_3 \cdot C_{0,3} = -1$, and $D_5 \cdot C_{0,3} = 1$, with all remaining intersection numbers equal to zero.

In particular, we obtain the coordinates for $C_{0,3}$ in the dual basis $N_1(Y_2)_{\mathbb{R}}$:

$$[C_{0,3}] = -[D_3]^\vee + [D_4]^\vee + [D_5]^\vee = (-1, 1, 1).$$

Since the other intersection calculations are likewise a matter of writing integer linear dependencies in $\mathbb{R}^3$, we do not reproduce each calculation. Further examples of this type of calculation will be given in Appendix 4.5. We nevertheless record from the above inequalities the coordinates of the generators of $N_1(Y_2)_{\mathbb{R}}$ with respect to the dual basis $\{[D_3]^\vee, [D_4]^\vee, [D_5]^\vee\}$:

$$\begin{aligned}
&[C_{0,1}] = [C_{1,2}] = [C_{1,4}] = [C_{2,5}] = (0, 1, 0), \\
&[C_{0,2}] = (0, 0, 1) \\
&[C_{2,4}] = (0, -1, 1) \\
&[C_{0,3}] = [C_{3,4}] = [C_{3,5}] = (-1, 1, 1), \\
&[C_{0,5}] = [C_{4,5}] = (1, 0, -1), \\
&[C_{0,4}] = (1, -1, 0).
\end{aligned}$$

By use of PORTA ([12]), the nef cone can be given as

$$\mathrm{Nef}(Y_2) = \langle [D_3] + [D_5], 2[D_3] + [D_4] + [D_5], [D_3] + [D_4] + [D_5] \rangle_{\geq 0}.$$

We denote the three extremal rays by N_1, N_2, and N_3, respectively. To calculate all pairs of intersections $N_i \cdot N_j$, we first calculate $D_r \cdot D_s$ for $r, s = 3, 4, 5$. For self-intersections, we rewrite the divisor using the relations in $N^1(Y_2)_{\mathbb{R}}$ to make the intersection transverse. For example,

$$[D_3]^2 = [D_3] \cdot ([D_0] - [D_4] - [D_5]) = [C_{0,3}] - [C_{3,4}] - [C_{3,5}].$$

In coordinates of the dual basis $\{[D_3]^\vee, [D_4]^\vee, [D_5]^\vee\}$, we obtain $[D_3]^2 = (1, -1, -1)$.

The other intersections are obtained analogously:

$$\begin{aligned}
[D_4]^2 &= (1,-2,0),\\
[D_5]^2 &= (1,-1,-1),\\
[D_3]\cdot[D_4] &= (-1,1,1),\\
[D_3]\cdot[D_5] &= (-1,1,1),\\
[D_4]\cdot[D_5] &= (1,0,-1).
\end{aligned}$$

Finally, we calculate the generators $[N_i]\cdot[N_j]$, $1 \le i \le j \le 3$, in the dual basis $N_1(Y_2)_{\mathbb{R}}$ by using the above intersections among the basis elements of $N^1(Y_2)_{\mathbb{R}}$:

$$\begin{aligned}
[N_1]^2 &= ([D_3]+[D_5])^2 = (0,0,0),\\
[N_2]^2 &= (2[D_3]+[D_4]+[D_5])^2 = (0,1,1),\\
[N_3]^2 &= ([D_3]+[D_4]+[D_5])^2 = (1,0,0),\\
[N_1]\cdot[N_2] &= ([D_3]+[D_5])\cdot(2[D_3]+[D_4]+[D_5]) = (0,1,0),\\
[N_1]\cdot[N_3] &= ([D_3]+[D_5])\cdot([D_3]+[D_4]+[D_5]) = (0,1,0),\\
[N_2]\cdot[N_3] &= (2[D_3]+[D_4]+[D_5])\cdot([D_3]+[D_4]+[D_5]) = (0,1,1).
\end{aligned}$$

Since the extremal ray $(0,0,1)$ of $\overline{\mathrm{Mov}}(Y_2)$ does not appear among the generators of $\mathcal{CI}(Y_2)$, it follows that $\mathcal{CI}(Y_2) \subsetneq \overline{\mathrm{Mov}}(Y_2)$.

4.5 Appendix: Intersections on the toric varieties X_r

We calculate the inqualities determining the nef and moving cones of X_r, $1 \le r \le 4$ given in Proposition 4.3.4. For $X_4 = \overline{L}_4$ we labeled the fan in Chapter 3 with a view towards the correspondence between boundary divisors of $\overline{L}_4$ and $\overline{M}_{0,6}$ given in Corollary 3.3.14. We will label the fans of the X_r, $r = 1,2,3$, so that the notation in the intersection calculations is least cumbersome. We refer to Section 3.2 for the basics of toric blow-ups. In particular, in figures below we show only the rays of the fans for the varieties, X_r; the fan structure is determined by the subdivisions as described for toric blow-ups in Section 3.2.

4.5.1 Inequalities for X_1

For the simplest example of X_1, or $\mathbb{P}^3$ blown up at a point, we will both calculate the inequalities of Proposition 4.3.4 and implement by hand Algorithm 4.4.3.

The fan of X_1 is depicted in Figure 4.2. The coordinates of the primitive generators of the rays of the fan are

$$u_0 = \begin{pmatrix} -1\\ -1\\ -1 \end{pmatrix}, u_1 = \begin{pmatrix} 1\\ 0\\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0\\ 1\\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0\\ 0\\ 1 \end{pmatrix}, v_1 = \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix},$$

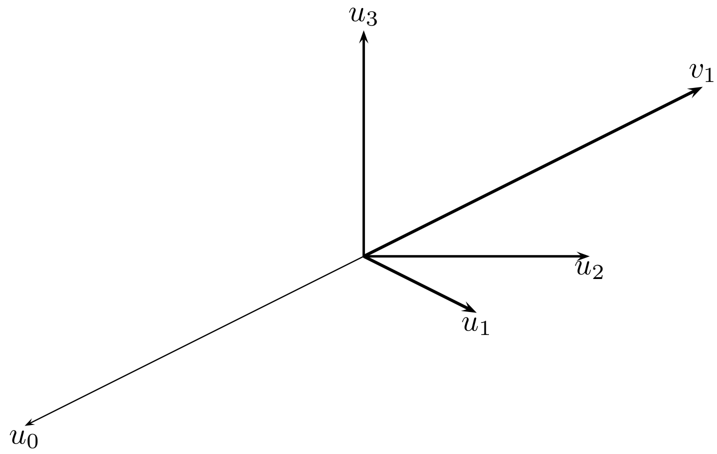


Figure 4.2: Rays of the fan of X_1

while the fan structure is determined by the description of toric blow-ups (see Section 3.2).

We label the divisors corresponding to u_i, $0 \le i \le 3$, by D_i, and the divisor corresponding to v_1 by E_1. For the relations in $N^1(X_1)_{\mathbb{R}}$, as in Example 4.4.5 it suffices to take principal divisors whose characters correspond to the coordinate functions on $N \cong \mathbb{Z}^3$, giving:

$$\begin{aligned} -[D_0] + [D_1] + [E_1] &= 0, \\ -[D_0] + [D_2] + [E_1] &= 0, \\ -[D_0] + [D_3] + [E_1] &= 0. \end{aligned}$$

The pseudoeffective cone is easily seen to be

$$\overline{\mathrm{Eff}}(X_1) = \langle [E_1], [D_1] \rangle_{\ge 0},$$

or, recalling the discussion about proper transforms from Section 3.2, in the Kapranov basis as

$$\overline{\mathrm{Eff}}(X_1) = \langle [E_1], [H] - [E_1] \rangle_{\ge 0},$$

thus giving the defining inequalities for $\overline{\mathrm{Mov}}(X_1)$ of Proposition 4.3.4, (i), or, equivalently,

$$\overline{\mathrm{Mov}}(X_1) = \langle [H]^\vee, [H]^\vee + [E_1]^\vee \rangle_{\ge 0}.$$

To determine $\mathrm{Nef}(X_1)$, we intersect an arbitrary divisor class $[D] = d_h[H] + d_1[E_1]$ with all one-strata of X_1. For example, to intersect the one-stratum $D_1 \cap E_1 = V(\langle u_1, v_1 \rangle_{\ge 0})$ with $[D]$, we read off the coefficients of the linear dependence relation among the vectors u_2, u_1, v_1, and u_3 such that the coefficients of u_2 and u_3 are positive (these vectors are generators for the two full-dimensional cones containing $\langle u_1, v_1 \rangle_{\ge 0}$; see Example 4.4.5 and Section 6.2 of [13] for further discussion). The linear relation is

$$(1)\begin{pmatrix}0\\1\\0\end{pmatrix}+(1)\begin{pmatrix}0\\0\\0\end{pmatrix}+(-1)\begin{pmatrix}1\\1\\1\end{pmatrix}+(1)\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},$$

so $(D_1 \cdot E_1) \cdot E_1 = -1$.

For rays that are not in a full-dimensional cone containing $\langle u_1, v_1\rangle_{\geq 0}$, the intersection with the corresponding divisor is 0, since such a divisor is disjoint from the one-stratum $D_1 \cap E_1$. It follows that $[D_1 \cap E_1] \cdot [H] = [D_1 \cap E_1] \cdot [D_0] = 0$, thus giving the inequality

$$[D] \cdot [D_1 \cap E_1] = -d_1 \geq 0.$$

By symmetry it is easy to see that

$$[D] \cdot [D_1 \cap E_1] = [D] \cdot [D_2 \cap E_1] = [D] \cdot [D_3 \cap E_1] = -d_1 \geq 0.$$

In terms of the dual Kapranov basis, it follows that

$$[D_1 \cap E_1] = [D_2 \cap E_1] = [D_3 \cap E_1] = [E_1]^\vee.$$

Likewise, the intersection of $D_1 \cap D_2 = V(\langle u_1, u_2\rangle_{\geq 0})$ with D is given by the linear dependence of u_0, u_1, u_2, and v_1:

$$(1)\begin{pmatrix}-1\\-1\\-1\end{pmatrix}+(0)\begin{pmatrix}1\\0\\0\end{pmatrix}+(0)\begin{pmatrix}0\\1\\0\end{pmatrix}+(1)\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},$$

so as above, $[D_1 \cap D_2] \cdot [H] = [D_1 \cap D_2] \cdot D_0 = 1$ and $[D_1 \cap D_2] \cdot [E_1] = 1$, giving the inequality

$$[D] \cdot [D_1 \cap D_2] = d_h + d_1 \geq 0.$$

By symmetry,

$$[D] \cdot [D_1 \cap D_2] = [D] \cdot [D_1 \cap D_3] = [D] \cdot [D_2 \cap D_3] = d_h + d_1 \geq 0,$$

and so

$$[D_1 \cap D_2] = [D_1 \cap D_3] = [D_2 \cap D_3] = [H]^\vee + [E_1]^\vee.$$

It remains to intersect $[D]$ with the one-cycle classes $[D_0 \cap D_i]$, $i = 1, 2, 3$. For $i = 1$ we obtain the relation

$$(1)\begin{pmatrix}0\\1\\0\end{pmatrix}+(1)\begin{pmatrix}-1\\-1\\-1\end{pmatrix}+(1)\begin{pmatrix}1\\0\\0\end{pmatrix}+(1)\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},$$

therefore $[D_0 \cap D_1] \cdot [D_0] = [D_0 \cap D_1] \cdot [D_1] = [D_0 \cap D_1] \cdot [D_2] = [D_0 \cap D_1] \cdot [D_3] = 1$, with all other intersections zero. If follows that $[D] \cdot [D_0 \cap D_1] = d_h$, and by symmetry, $[D] \cdot [D_0 \cap D_1] = [D] \cdot [D_0 \cap D_2] = [D] \cdot [D_0 \cap D_3]$. This inequality, however, equals the sum of the other two, so it does not contribute an extremal ray to $\mathrm{Nef}(X_1)$.

Therefore the nef cone of X_1 is

$$\begin{aligned}\mathrm{Nef}(X_1) &= \{d_h[H] + d_1[E_1] : -d_1 \geq 0,\, d_h + d_1 \geq 0\}\\ &= \langle [H], [H] - [E_1]\rangle_{\geq 0}.\end{aligned}$$

Finally, the intersections of elements in the Kapranov basis are as follows:

$$\begin{aligned}[H]^2 &= [H]^\vee,\\ [H]\cdot[E_1] &= 0,\\ [E_1]\cdot[E_1] &= [E_1]^2 = [E_1]^\vee.\end{aligned}$$

The first two equalities follows from the projection formula, as in the proof of Proposition 4.3.5. The third can be calculated by rewriting one of the $[E_1]$ so that the resulting intersection is transverse, for example, $[E_1] = [D_0] - [D_1] = [H] - [D_1]$. By the above calculation for intersections with $D_1 \cap E_1$, the result follows.

We now implement Algorithm 4.4.3 for X_1. The complete intersection cone is

$$\begin{aligned}\mathcal{CI}(X_1) &= \langle [H]^2, [H]\cdot([H]-[E_1]), ([H]-[E_1])^2\rangle_{\geq 0}\\ &= \langle [H]^\vee, [H]^\vee + [E_1]^\vee\rangle_{\geq 0},\end{aligned}$$

which is what we obtained above for the movable cone, $\overline{\mathrm{Mov}}(X_1)$.

4.5.2 Inequalities for X_2

Rather than blowing up $[1,0,0,0]$, $[0,1,0,0]$, and the proper transform of the lines joining these points, we instead blow up $[1,0,0,0]$, $[0,0,0,1]$, and the proper transform of the line joining them, since the resulting fan—whose rays are shown in Figure 4.3—is more symmetric, and more amenable to calculations.

The generators of the rays of X_2 are

$$u_0 = \begin{pmatrix}-1\\-1\\-1\end{pmatrix}, u_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, u_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, u_3 = \begin{pmatrix}0\\0\\1\end{pmatrix},$$
$$v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, v_2 = \begin{pmatrix}0\\0\\-1\end{pmatrix}, v_{12} = \begin{pmatrix}1\\1\\0\end{pmatrix}.$$

We label the divisors as $D_i = V(u_i)$, $i = 0,\ldots,3$, $E_i = V(v_j)$, $j = 1,2$, and $E_{12} =$

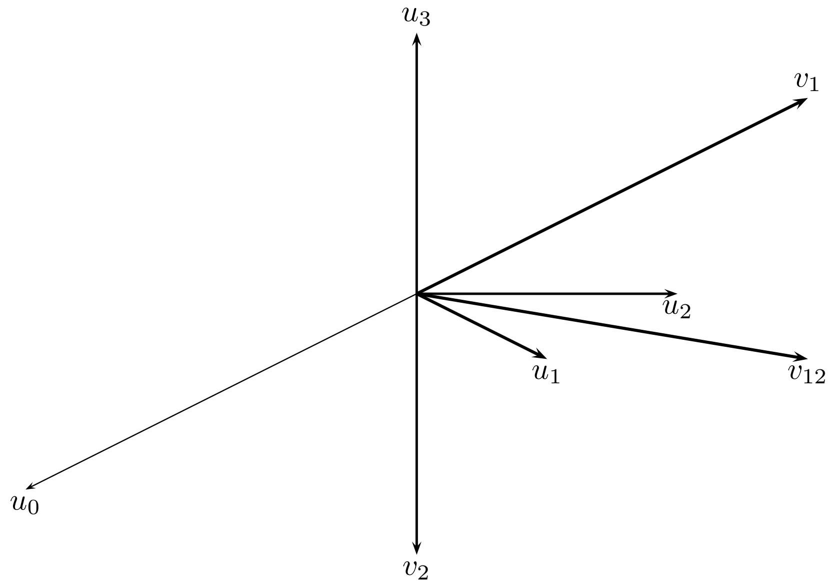


Figure 4.3: Rays of the fan of X_2

$V(v_{12})$. The relations in $N^1(X_2)_{\mathbb{R}}$ are generated by

$$-[D_0]+[D_1]+[E_1]+[E_{12}]=0,$$
$$-[D_0]+[D_2]+[E_1]+[E_{12}]=0,$$
$$-[D_0]+[D_3]+[E_1]-[E_2]=0,$$

so the pseudoeffective cone is

$$\overline{\mathrm{Eff}}(X_2)=\langle [E_1],[E_2],[E_{12}],[H]-[E_1]-[E_2]-[E_{12}]\rangle_{\geq 0}.$$

Intersecting each generator with a one-cycle class $[C]=c_h[H]^\vee+c_1[E_1]^\vee+c_2[E_2]^\vee+c_{12}[E_{12}]^\vee \in N_1(X_2)_{\mathbb{R}}$ written in the dual Kapranov basis gives the inequalities for $\overline{\mathrm{Mov}}(X_2)$ of Proposition 4.3.4 (ii) just as in the proof of Proposition 4.3.3.

For the defining inequalities of the nef cone of X_2, we calculate one of each type of inequality from Proposition 4.3.4 (ii). The others follow by symmetry, or can be obtained as sums of the inequalities already given.

The intersections with the one-stratum $D_1\cap E_1 = V(\langle u_1,v_1\rangle_{\geq 0})$ are determined by the relation among u_3, u_1, v_1, and v_{12},

$$(1)\begin{pmatrix}0\\0\\1\end{pmatrix}+(0)\begin{pmatrix}1\\0\\0\end{pmatrix}+(-1)\begin{pmatrix}1\\1\\1\end{pmatrix}+(1)\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Hence $[D_1\cap E_1]\cdot[H]=[D_1\cap E_1]\cdot[D_3+E_1]=0$, $[D_1\cap E_1]\cdot[E_1]=-1$, $[D_1\cap E_1]\cdot[E_2]=0$, and $[D_1 \cap E_1] \cdot [E_{12}] = 1$, which gives the inequality

$$[D] \cdot [D_1 \cap E_1] = -d_1 + d_{12} \geq 0.$$

By a direct check or by the symmetry of the fan, we obtain as well

$$[D] \cdot [D_i \cap E_1] = -d_i + d_{12} \geq 0,$$

for $i = 1, 2$, and so $[D_i \cap E_1] = -[E_i]^\vee + [E_{12}]^\vee$ in the dual basis $N_1(X_2)_\mathbb{R}$.

Intersections with $D_1 \cap E_{12} = V(\langle u_1, v_{12}\rangle_{\geq 0})$ are determined by the relation among v_1, u_1, v_{12}, and v_2,

$$(1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (1)\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus $[D_1 \cap E_{12}] \cdot [H] = [D_1 \cap E_{12}] \cdot ([D_3] + [E_1]) = 1$, $[D_1 \cap E_{12}] \cdot [E_1] = 1$, $[D_1 \cap E_{12}] \cdot [E_2] = 1$, and $[D_1 \cap E_{12}] \cdot [E_{12}] = -1$, giving

$$[D] \cdot [D_1 \cap E_{12}] = d_h + d_1 + d_2 - d_{12} \geq 0.$$

Symmetry implies that $[D] \cdot [D_1 \cap E_{12}] = [D] \cdot [D_2 \cap E_{12}]$, and so in terms of the dual basis,

$$[D_1 \cap E_{12}] = [D_2 \cap E_{12}] = [H]^\vee + [E_1]^\vee + [E_2]^\vee - [E_{12}]^\vee.$$

The final inequality given in Proposition 4.3.4 (ii) arises from intersecting $[D]$ with $E_1 \cap E_{12} = V(\langle v_1, v_{12}\rangle_{\geq 0})$. From the relation among u_1, v_1, v_{12}, and u_2,

$$(1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (0)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

it follows that $[E_1 \cap E_{12}] \cdot [H] = [E_1 \cap E_{12}] \cdot ([D_3] + [E_1]) = 0$, $[E_1 \cap E_{12}] \cdot [E_1] = 0$, $(E_1 \cdot E_{12}) \cdot E_2 = 0$, and $(E_1 \cdot E_{12}) \cdot E_{12} = -1$, hence

$$[D] \cdot [E_1 \cap E_{12}] = -d_{12} \geq 0,$$

and as before, $[E_1 \cap E_{12}] = [E_2 \cap E_{12}] = -[E_{12}]^\vee$.

The remaining intersections with one-cycles of X_2 are easily checked to be either zero, or obtainable as effective sums of intersections already calculated.

In the course of determining $\mathrm{Nef}(X_3)$, we have already calculated most intersectios of pairs of divisor classes from the Kapranov basis. For $i = 1, 2$,

$$\begin{aligned}
&[H]^2 = [H]^\vee,\\
&[H] \cdot [E_i] = 0\\
&[H] \cdot [E_{12}] = [E_i] \cdot [E_{12}] = -[E_{12}]^\vee,\\
&[E_{12}]^2 = 2[E_{12}]^\vee - [H]^\vee - [E_1]^\vee - [E_2]^\vee.
\end{aligned}$$

The first and second equalities follow from the projection formula as in the proof of Proposition 4.3.5. The third equality was established by intersections with the one-stratum $E_1 \cap E_{12}$.

The final equality can be obtained by substituting $[E_{12}] = [D_0] - [D_1] - [E_1]$. Since $[E_{12}] \cdot [D_0] = 0$, our calculations involving $D_1 \cap E_{12}$ and $E_1 \cap E_{12}$ yield the last equality.

4.5.3 Inequalities for X_3

As with X_2, the fan of X_3 is nicer to work with for a certain choice of the three points, namely, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, and $[0, 0, 0, 1]$, and the proper transforms of the lines spanned by these points. The rays of the resulting fan of X_3 is shown in Figure 4.4. Note that the dashed lines are merely to indicate depth, and do not describe the fan structure.

The coordinates of the primitive generators for the rays of X_3 are

$$u_0 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$v_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},$$

$$v_{12} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, v_{13} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, v_{23} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}.$$

We denote the boundary divisors by $D_i = V(u_i)$, $i = 0, \ldots, 3$, $E_j = V(v_j)$, $j = 1, 2, 3$, and $E_{kl} = V(v_{kl})$, $1 \le k < l \le 3$. The relations in $N^1(X_3)_{\mathbb{R}}$ are generated by

$$\begin{aligned}
-[D_0] + [D_1] - [E_1] - [E_{12}] - [E_{13}] = 0,\\
-[D_0] + [D_2] - [E_2] - [E_{12}] - [E_{23}] = 0,\\
-[D_0] + [D_3] - [E_3] - [E_{13}] - [E_{23}] = 0,
\end{aligned}$$

so the pseudoeffective cone is given by

$$\begin{aligned}
\overline{\mathrm{Eff}}(X_3) = \langle &[E_1], [E_2], [E_3], [E_{12}], [E_{13}], [E_{23}],\\
&[H] - [E_1] - [E_2] - [E_3] - [E_{12}] - [E_{13}] - [E_{23}]\rangle_{\ge 0}.
\end{aligned}$$

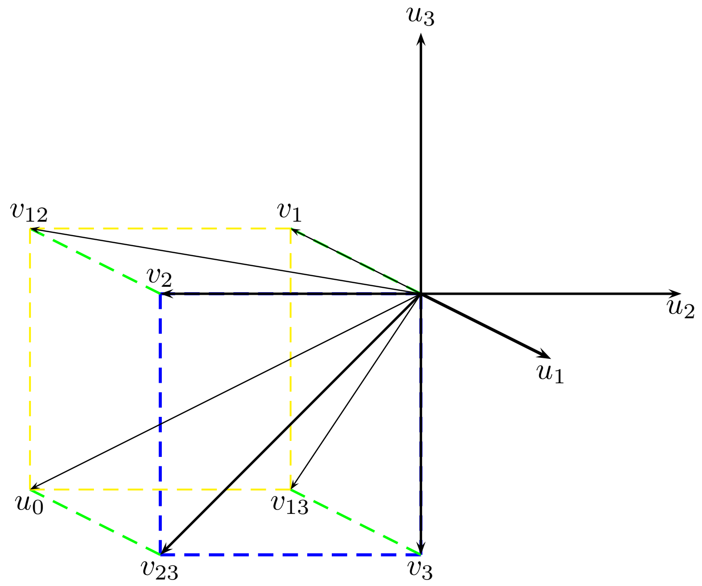


Figure 4.4: Rays of the fan of X_3

Intersecting each generator with a one-cycle class $[C] \in N_1(X_3)_{\mathbb{R}}$ gives the inequalities for $\overline{\mathrm{Mov}}(X_3)$ of Proposition 4.3.4 (iii).

As above, we calculate one of each of the types of intersections determining the nef cone of X_3, as the other inqualities follow by symmetry, or are sums of the ones already calculated.

For the first equality, consider the one-stratum $E_1 \cap E_{12} = V(\langle v_1, v_{12}\rangle_{\geq 0})$. The relation among the ray generators u_0, v_1, v_{12}, and u_3 is

$$(1)\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + (0)\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + (1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

thus determining the intersections with $E_1 \cap E_{12}$: $[E_1 \cap E_{12}] \cdot [E_i] = 0$ for $i = 1, 2, 3$, $[E_1 \cap E_{12}] \cdot [E_{jk}] = -1$ if $j, k = 1, 2$, and 0 otherwise, and $[E_1 \cap E_{12}] \cdot [H] = [E_1 \cap E_{12}] \cdot ([D_3] + [E_1] + [E_2] + [E_{12}]) = 0$. The resulting inequality is

$$[D] \cdot [E_1 \cap E_{12}] = -d_{12} \geq 0.$$

The symmetry of the fan implies that

$$[D] \cdot [E_i \cdot E_{ij}] = -d_{ij} \geq 0,$$

for $1 \leq i < j \leq 3$, or, in terms of the dual basis, $[E_i] \cdot [E_{ij}] = -[E_{ij}]^\vee$.

Next consider $D_0 \cap E_{12} = V(\langle u_0, v_{12}\rangle_{\geq 0})$. The relation among v_1, u_0, v_{12}, and v_2,

$$(1)\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + (0)\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + (-1)\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + (1)\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

yields $[D_0 \cap E_{12}] \cdot [E_1] = [D_0 \cap E_{12}] \cdot [E_2] = 1$, $[D_0 \cap E_{12}] \cdot [E_{12}] = -1$, $[D_0 \cap E_{12}] \cdot [H] = [D_0 \cap E_{12}] \cdot ([D_3] + [E_1] + [E_2] + [E_{12}]) = 1$, with all other intersections equal to zero, hence,

$$[D] \cdot [D_0 \cap E_{12}] = d_h + d_1 + d_2 - d_{12} \geq 0,$$

and by symmetry,

$$[D_0 \cap E_{ij}] = [H]^\vee + [E_i]^\vee + [E_j]^\vee - [E_{ij}]^\vee,$$

for $1 \leq i < j \leq 3$.

Finally, intersections with $D_0 \cap E_1 = V(\langle u_0, v_1\rangle_{\geq 0})$ can be read off of the relation among v_{12}, u_0, v_1, and v_{13},

$$(1)\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + (-1)\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + (1)\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows that $[D_0 \cap E_1 \cdot [E_1] = -1$, $[D_0 \cap E_1] \cdot [E_{12}] = [D_0 \cap E_1] \cdot [E_{13}] = 0$, with all other intersections equal to zero, thus determining the inequality

$$[D] \cdot [D_0 \cap E_1] = -d_1 + d_{12} + d_{13} \geq 0,$$

and, by symmetry, for $1 \leq i \leq 3$, $\{i, j, k\} = \{1, 2, 3\}$,

$$[D] \cdot [D_0 \cap E_i] = -d_i + d_{ij} + d_{ik} \geq 0.$$

In the dual basis we obtain $[D_0 \cap E_i] = -[E_i]^\vee + [E_{ij}]^\vee + [E_{ik}]^\vee$.

Intersections with other one-strata are easily seen to be obtainable as effective sums of the intersections already calculated.

For $1 \leq i \leq 3$ and $1 \leq j < k \leq 3$, the intersections of elements of the Kapranov basis of X_3 in terms of the dual basis are

$$\begin{aligned}
&[H]^2 = [H]^\vee,\\
&[H]\cdot[E_i] = 0,\\
&[H]\cdot[E_{jk}] = [E_j]\cdot[E_{jk}] = [E_k]\cdot[E_{jk}] = -[E_{12}]^\vee,\\
&[E_i]^2 = [E_i]^\vee,\\
&[E_j]\cdot[E_k] = 0,\\
&[E_{jk}]^2 = 2[E_{jk}]^\vee - [H]^\vee - [E_j]^\vee - [E_k]^\vee,\\
&[E_{jk}]\cdot[E_{j'k'}] = 0,
\end{aligned}$$

where in the last equality $j' < k'$ are elements of $\{1,\dots,3\}$ such that $\{j,k\} \neq \{j',k'\}$. The first and second equalities result from the projection formula, as with X_2 and in the proof of Proposition 4.3.3. The third set of equalities follow by noting that $[H] = [D_3]+[E_1]+[E_2]+[E_{12}]$, and then using the intersection calculations for $[E_1\cdot E_{12}]$ above.

The equality $[E_i]^2 = [E_i]^\vee$ can be proved by rewriting $[E_i]$ so that the intersection is transverse. For example, $[E_1] = -[D_0]+[D_1]-[E_{12}]-[E_{13}]$, so that $[E_1]^2 = -[E_1]\cdot([D_0]+[E_{12}]+[E_{13}]) = [E_1]^\vee$.

The equality $[E_j]\cdot[E_k] = 0$ holds since the divisors E_j and E_k are disjoint, as are E_{jk} and $E_{j'k'}$ for $\{j,k\} \neq \{j',k'\}$ chosen as above, hence $[E_{jk}]\cdot[E_{j'k'}] = 0$.

It remains to prove the penultimate equality above. By symmetry it suffices to determine $[E_{12}]^2$. We can make this intersection transverse by rewriting it as $[E_{12}]\cdot(-[D_0]+[D_1]-[E_1]-[E_{13}]) = [E_{12}]\cdot(-[D_0]-[E_1])$. By the above calculations, we obtain the desired result.

4.5.4 Inequalities for $X_4 = \overline{L}_4$

We revert for $X_4 = \overline{L}_4$ to the labeling of rays as in Figure 3.4. Recall that the coordinates of the generating vectors of the rays are

$$v_{234} = \begin{pmatrix} -1\\ -1\\ -1 \end{pmatrix}, v_{134} = \begin{pmatrix} 1\\ 0\\ 0 \end{pmatrix}, v_{124} = \begin{pmatrix} 0\\ 1\\ 0 \end{pmatrix}, v_{123} = \begin{pmatrix} 0\\ 0\\ 1 \end{pmatrix},$$

$$v_1 = \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1\\ 0\\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0\\ -1\\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0\\ 0\\ -1 \end{pmatrix},$$

$$v_{12} = \begin{pmatrix} 0\\ 1\\ 1 \end{pmatrix}, v_{13} = \begin{pmatrix} 1\\ 0\\ 1 \end{pmatrix}, v_{14} = \begin{pmatrix} 1\\ 1\\ 0 \end{pmatrix},$$

$$v_{23} = \begin{pmatrix} -1\\ -1\\ 0 \end{pmatrix} . v_{24} = \begin{pmatrix} -1\\ 0\\ -1 \end{pmatrix}, v_{34} = \begin{pmatrix} 0\\ -1\\ -1 \end{pmatrix}.$$

We label the divisors to facilitate calculations in the Kapranov basis. For $1 \leq i \leq 4$ we write $E_i = V(\langle v_i \rangle_{\geq 0})$, and for $1 \leq j < k \leq 4$ we write $E_{jk} = V(\langle v_{jk} \rangle_{\geq 0})$. For the remaining boundary divisors we write $D_{ijk} = V(\langle v_{ijk} \rangle_{\geq 0}$ for $1 \leq i < j < k \leq 4$. Note that D_{ijk} are the proper transforms of the plane containing the points l_i, l_j, and l_k of $\mathbb{P}^3$ (recall the labeling convention of Notation 3.2.5).

The relations in $N^1(X_4)_{\mathbb{R}}$ are generated by

$$[E_1] + [E_{13}] + [E_{14}] + [D_{134}] - [E_2] - [E_{23}] - [E_{24}] - [D_{234}] = 0,$$
$$[E_1] + [E_{12}] + [E_{14}] + [D_{124}] - [E_3] - [E_{23}] - [E_{34}] - [D_{234}] = 0,$$
$$[E_1] + [E_{12}] + [E_{13}] + [D_{123}] - [E_4] - [E_{24}] - [E_{34}] - [D_{234}] = 0.$$

The pseudoeffective cone of X_4 is

$$\overline{\mathrm{Eff}}(X_4) = \langle [E_i], [E_{jk}], [D_{lmn}] : 1 \leq i \leq 4,\ 1 \leq j < k \leq 4,\ 1 \leq l < m < n \leq 4 \rangle_{\geq 0},$$

thus giving the inequalities for $\overline{\mathrm{Mov}}(X_4)$ from Proposition 4.3.4 (iv).

We give as before one calculation for each type of defining inequality for the nef cone of X_4. For the first inequality, consider $E_1 \cap E_{12}$, and the corresponding relation among v_{123}, v_1, v_{12}, and v_{124}:

$$(1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (0)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence $[E_1 \cap E_{12}] \cdot [E_1] = 0$, $[E_1 \cap E_{12}] \cdot [E_{12}] = -1$, and $[E_1 \cap E_{12}] \cdot [H] = [E_1 \cap E_{12}] \cdot ([D_{234}] + [E_2] + [E_3] + [E_4] + [E_{23}] + [E_{24}] + [E_{34}]) = 0$, with all other intersections equal to zero. We obtain the inequality

$$[D] \cdot [E_1 \cap E_{12}] = -d_{12} \geq 0,$$

By the symmetry of the fan, we obtain additionally $-d_{ij} \geq 0$ for $1 \leq i < j \leq 4$, and, in terms of the dual basis,

$$[E_i \cap E_{ij}] = -[E_{ij}]^\vee.$$

Next consider $D_{123} \cap E_{12} = V(\langle v_{123}, v_{12} \rangle_{\geq 0}$. The relation among vectors v_1, v_{123}, v_{12}, and v_2 is

$$(1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (1)\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

hence $[D_{123} \cap E_{12}] \cdot [E_1] = [D_{123} \cap E_{12}] \cdot [E_2] = 1$, $[D_{123} \cap E_{12}] \cdot [E_{12}] = -1$, and $[D_{123} \cap E_{12}] \cdot [H] = [D_{123} \cap E_{12}] \cdot ([D_{134}] + [E_1] + [E_3] + [E_4] + [E_{13}] + [E_{14}] + [E_{34}]) = 1$, giving the inequality

$$[D] \cdot [D_{123} \cap E_{12}] = d_h + d_1 + d_2 - d_{12} \geq 0.$$

Thus for distinct elements i, j of $\{1, \ldots, 4\}$, we obtain the inequalities $d_h + d_i + d_j - d_{ij}$, and for $k \in \{1, \ldots, 4\}$ distinct from i and j, we have the following expression in the dual basis:

$$[D_{ijk} \cap E_{ij}] = [H]^\vee + [E_i]^\vee + [E_j]^\vee + [E_{ij}]^\vee.$$

For the final type of inequality, consider $D_{123} \cap E_1 = V(\langle v_{123}, v_1 \rangle_{\geq 0})$. The relation among v_{12}, v_{123}, v_1, and v_{13} is

$$(1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (1)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

thus giving the intersections $[D_{123} \cap E_1] \cdot [E_1] = -1$, $[D_{123} \cap E_1] \cdot [E_{12}] = [D_{123} \cap E_1] \cdot [E_{13}] = 1$, $[D_{123} \cap E_1] \cdot [H] = [D_{123} \cap E_1] \cdot ([D_{234}] + [E_2] + [E_3] + [E_4] + [E_{23}] + [E_{24}] + [E_{34}]) = 0$, with all other intersections equal to zero, therefore

$$[D] \cdot [D_{123} \cap E_1] = -d_1 + d_{12} + d_{13} \geq 0.$$

By symmetry, we obtain both the final type of equality from Proposition 4.3.4 (*iv*) and the expressions in $N_1(X_4)$,

$$[D_{ijk} \cap E_i] = -[E_i]^\vee + [E_{ij}]^\vee + [E_{ik}]^\vee$$

for distinct $i, j, k \in \{1, \ldots, 4\}$.

To conclude, we record the intersections of divisor classes in the Kapranov basis of $X_4 = \overline{L}_4$: , for $i = 1, \ldots, 4$, and $1 \leq j < k \leq 4$,

$$\begin{aligned}
&[H]^2 = [H]^\vee, \\
&[H] \cdot [E_i] = 0 \\
&[H] \cdot [E_{jk}] = [E_j] \cdot [E_{jk}] = [E_k] \cdot [E_{jk}] = -[E_{12}]^\vee, \\
&[E_i]^2 = [E_i]^\vee, \\
&[E_j] \cdot [E_k] = 0 \\
&[E_{jk}]^2 = 2[E_{jk}]^\vee - [H]^\vee - [E_j]^\vee - [E_k]^\vee, \\
&[E_{jk}] \cdot [E_{j'k'}] = 0,
\end{aligned}$$

where in the last equality $1 \leq j' < k' \leq 4$ are chosen such that $\{j, k\} \neq \{j', k'\}$.

The proofs of these equalities is analogous to the statements for the other X_r. The first two equalities can be proved by the projection formula, while the third is a result of the above intersection calculations with $[E_j \cap E_{jk}]$ for $j < k$ elements of $\{1, \ldots, 4\}$, plus a substitution for $[H]$ as above.

To prove $[E_i]^2 = [E_i]^\vee$, it suffices to show $[E_1]^2 = [E_1]^\vee$. We express $[E_1]^2$ as $[E_1] \cdot ([E_2] + [E_{23}] + [E_{24}] + [D_{234}] - [E_{13}] - [E_{14}] - [D_{134}]) = -[E_1] \cdot ([E_{13}] + [E_{14}] + [D_{134}])$, and apply the intersection calculations above to obtain the result.

Similarly we prove the second to last equality by rewriting $[E_{jk}]$. By symmetry, we consider $[E_{12}]^2 = [E_{12}] \cdot ([E_3] + [E_{23}] + [E_{34}] + [D_{234}] - [E_1] - [E_{14}] - [D_{124}]) = -[E_{12}] \cdot ([E_1] + [D_{124}]) = 2[E_{12}]^\vee - [H]^\vee - [E_1]^\vee - [E_2]^\vee$ by the previously obtained intersection calculations.

The final equality holds since the divisors E_{jk} and $E_{j'k'}$ are disjoint for $j < k$, $j' < k'$ with $\{j, k\} \neq \{j', k'\}$.

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 20.07.2010 Paul L. Larsen